

# Bifurcations in a Coupled Rössler System

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**SUMMARY** We propose an equivalent circuit model described by the Rössler equation. Then we can consider a coupled Rössler system with a physical meaning on the connection. We consider an oscillatory circuit such that two identical Rössler circuits are coupled by a resistor. We have studied three routes to entirely and almost synchronized chaotic attractors from phase-locked periodic oscillations. Moreover, to simplify understanding of synchronization phenomena in the coupled Rössler system, we investigate a mutually coupled map that shows analogous locking properties to the coupled Rössler system.

**Key words:** Rössler circuit, coupled oscillator, coupled map, nonlinear dynamical system

## 1. Introduction

The Rössler equation [1], [2] is known as an autonomous dynamical system to exhibit chaotic attractors. Because it has no physical model from which the equation derived, we first propose an equivalent circuit model described by the Rössler equation. Then we can consider a coupled Rössler system with a physical meaning on the connection. In this paper we consider an oscillatory circuit such that two identical Rössler circuits are coupled by a resistor. Many investigations have been done on synchronization of periodic and chaotic oscillations in mutually coupled oscillators [3]–[6]. The coupled Rössler circuit is useful to consider a physical mechanism of synchronized periodic motions, generation of chaos, synchronization of chaotic oscillations and so on. Moreover, in recent years, chaotic synchronization in a coupled discrete dynamical system is studied [7]. In this paper, to simplify understanding of synchronization in the coupled Rössler system, we also investigate a mutually coupled map that shows analogous locking phenomena to the coupled Rössler system. Our objective of the analysis is to answer the question, “How does chaotic synchronization generate through bifurcations of synchronized periodic oscillations?”

## 2. Coupled Rössler Circuit

The Rössler equation is a three-dimensional autonomous differential equation given by

$$\begin{aligned}\frac{dx}{dt} &= -y - z \\ \frac{dy}{dt} &= x + ay \\ \frac{dz}{dt} &= bx - cz + xz\end{aligned}\quad (1)$$

with the parameters  $a$ ,  $b$  and  $c$ . We now consider the circuit shown in Fig. 1. The dynamics is written as

$$\begin{aligned}C\frac{dv}{d\tau} &= -i_1 - i_2 \\ L_1\frac{di_1}{d\tau} &= v + ri_1 \\ L_2\frac{di_2}{d\tau} &= v - Ri_2 + e(v, i_2)\end{aligned}\quad (2)$$

where the controlled source  $e$  is characterized by  $e(v, i_2) = \alpha vi_2$  with a constant parameter  $\alpha$ . Rescaling coordinate system

$$\begin{aligned}x &= \sqrt{C}v, \quad y = \sqrt{L_1}i_1, \quad z = \sqrt{L_1}i_2, \\ t &= \frac{1}{\sqrt{L_1C}}\tau\end{aligned}$$

and changing parameters

$$\begin{aligned}a &= r\sqrt{\frac{C}{L_1}}, \quad b = \frac{L_1}{L_2}, \quad c = \frac{R}{L_2}\sqrt{L_1C}, \\ \beta &= \frac{\alpha\sqrt{L_1}}{L_2}\end{aligned}$$

lead to

$$\begin{aligned}\frac{dx}{dt} &= -y - z \\ \frac{dy}{dt} &= x + ay \\ \frac{dz}{dt} &= bx - cz + \beta xz\end{aligned}\quad (3)$$

Therefore, (2) with  $\beta = 1$  is equivalent to the Rössler equation. In this paper we will study a circuit such that two identical Rössler circuits are coupled by a conductor  $G$  through  $a$ - $a'$  port in Fig. 1. The dynamics is normalized as follows

$$\begin{aligned}\frac{dx_k}{dt} &= -y_k - z_k - d(x_k - x_{k-1}) \\ \frac{dy_k}{dt} &= x_k + ay_k\end{aligned}$$

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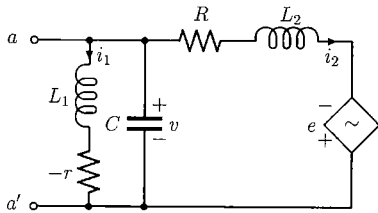


Fig. 1 Rössler circuit.

$$\frac{dz_k}{dt} = bx_k - cz_k + x_k z_k, \quad k = 1, 2, \quad x_0 \equiv x_2 \tag{4}$$

where

$$d = G \sqrt{\frac{L_1}{C}}$$

In the following, parameters except  $a$  and  $c$  are fixed as  $b = 1$  and  $d = 0.01$ .

### 3. Coupled Map

We treat a discrete dynamical system composed by coupling of two quadratic maps:

$$\begin{aligned} x(k+1) &= x^2(k) - \lambda - \delta(y(k) - x(k)) \\ y(k+1) &= y^2(k) - \lambda - \delta(x(k) - y(k)) \end{aligned} \tag{5}$$

or equivalently

$$\begin{aligned} u(k+1) &= \frac{1}{\sqrt{2}}(u^2(k) + v^2(k)) - \sqrt{2}\lambda \\ v(k+1) &= (\sqrt{2}u(k) + 2\delta)v(k) \end{aligned} \tag{6}$$

with an orthogonal transformation such that

$$\begin{aligned} u &= \frac{1}{\sqrt{2}}(x + y) \\ v &= \frac{1}{\sqrt{2}}(x - y) \end{aligned} \tag{7}$$

The coupling coefficient  $\delta$  is fixed as 0.01.

### 4. Some Preliminaries

We summarize notations about analysis of fixed or periodic points in (5) and limit cycles observed in (4). We discuss qualitative properties of a periodic solution of (4) by reducing to the study of a periodic point of the Poincaré map [8]–[10], defined by a cross section  $\{(U_1, U_2) \in R^6 \mid x_1 = 0, dx_1/dt < 0\}$ , where  $U_i = (x_i, y_i, z_i), i = 1, 2$ . We now define notations for fixed or periodic points of (5) and the Poincaré map for (4). The symbol  ${}_k D_l^m$  (resp.  ${}_k I_l^m$ ) denotes a hyperbolic periodic point such that  $D$  (resp.  $I$ ) indicates a type with even (resp. odd) number of characteristic multipliers on

the real axis  $(-\infty, -1)$ ,  $k$  indicates the number of characteristic multiplier outside the unit circle in the complex plane,  $m$  indicates  $m$ -periodic point, and  $l$  denotes the number to distinguish the same qualitative type of periodic points. Note that the system of (4) is invariant with respect to the replacing of state variables:

$$\begin{aligned} P_r &: R^6 \rightarrow R^6; \\ (U_1, U_2) &\mapsto P_r(U_1, U_2) = (U_2, U_1) \end{aligned} \tag{8}$$

and similarly (5) is invariant with respect to the transformation

$$\begin{aligned} P_m &: R^2 \rightarrow R^2; \\ (x, y) &\mapsto P_m(x, y) = (y, x) \end{aligned} \tag{9}$$


Therefore, there may exist periodic solutions with symmetrical properties. If there is another periodic orbit with a symmetrical property (appeared in the following section), then we add pre-superscripts  $*$  and  $**$  to symbols for their periodic points.

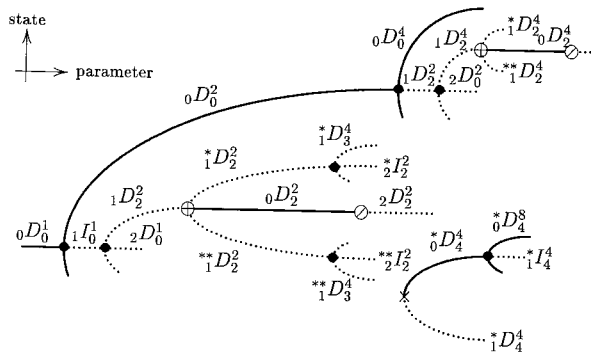
## 5. Main Results

### 5.1 Properties of Periodic Points in Coupled Map

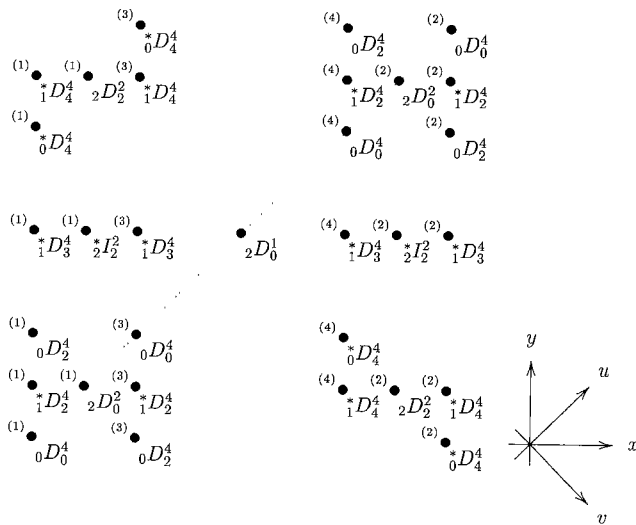
Before showing results on the coupled Rössler system, we show properties of periodic points in the weakly coupled map (5). By increasing the value of  $\lambda$ , one-parameter bifurcation diagram is schematically illustrated in Fig. 2. In the figure, a couple of periodic orbits specified by symbols with pre-superscripts  $*$  and  $**$  is symmetric each other with respect to the  $u$ -axis. Figure 3 shows a diagram of phase portrait for periodic points in (5) or (6) at  $\lambda = 1.3$ . By the iteration of the discrete system, the periodic point in Fig.3 moves in the increasing order of the parenthesized number. It is well known that, in a single quadratic map, the periodic points can be imbedded in a horseshoe rectangle region. In the two-coupled quadratic map, we see that the periodic points can be imbedded in the double horseshoe operations in both  $u$  and  $v$  directions. Here we define a symbolic rotation sequence [11]–[13] for an orbit on the  $v$ -axis or the orthogonal subspace of  $P_m$ -invariant subspace. For the rotation sequence, we use a notation written by the alphabet  $\{L, R\}$  with full bits [13], which is derived from the rotation sequence of Myrberg, e.g., the symbol [LRLR] is for  ${}_0^* D_4^4$  in Fig. 3.

### 5.2 Bifurcations in Coupled Rössler System

We shall show numerical results on bifurcational process to chaotic attractors. Figure 4 shows a bifurcation diagram to see a fundamental route to chaotic states. The periodic solution observed in the area by shading , is locked with in-phase mode. By increasing the value of  $a$  for fixed  $c = 3$ , we obtain the one-parameter bifurcation diagram, which is qualitatively same as the



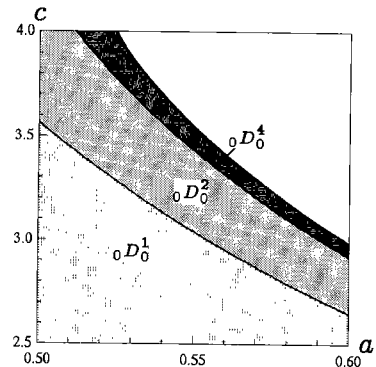
**Fig. 2** Schematic diagram of one-parameter bifurcation for (5) (resp. (4)), where the ordinate shows norms of the states and the abscissa shows the parameter  $\lambda$  (resp.  $a$ ) in (5) (resp. (4)). The solid and dotted lines denote stable and unstable periodic points, respectively. Symbols: ● for period-doubling bifurcation; × for tangent bifurcation; ⊕ for D-type of branching; and ⊗ for the Hopf bifurcation.



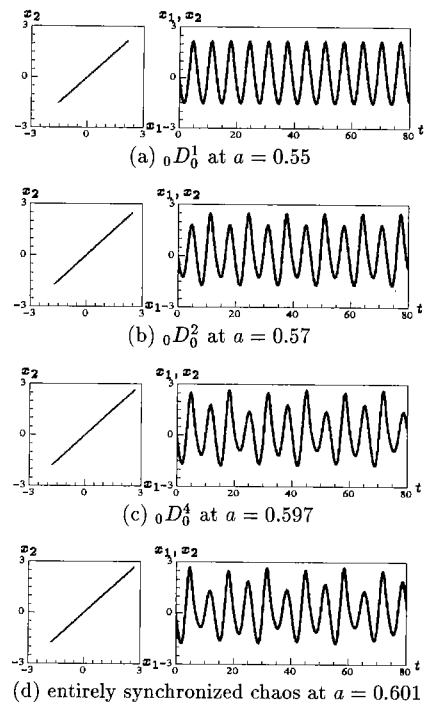
**Fig. 3** Schematically illustrated phase portrait of periodic points of (5) at  $\lambda = 1.3$ . The dotted line satisfies  $x = y$  and  $v = 0$ . The parenthesized number shows the order of iteration. The periodic points in their symmetrical positions with respect to the  $x = y$  line or the  $u$ -axis are omitted.

figure in Fig. 2. The symbols with pre-superscripts \* and \*\* in Fig. 2 specify the periodic points whose corresponding solutions of (4), say  $*U$  and  $**U$ , satisfy  $P_r(*U) = **U$ . We see that several routes to chaotic attractors are found: (i)  ${}_0D_0^{2^n}$ ,  $n = 0, 1, 2, \dots$ ; (ii)  ${}_0D_2^{2^n}$ ,  $n = 1, 2, \dots$ ; and (iii)  ${}_0D_4^{2^n}$ ,  $n = 2, 3, \dots$ . We now consider properties of the sequences. First, we consider the sequence of (i), caused by continuous variation of the parameter  $a$ . The fixed point  ${}_0D_0^1$  corresponds to the in-phase-locked periodic oscillation, and any  ${}_0D_0^{2^n}$  satisfies  $u_1(t) = u_2(t)$  for all  $t$ , where  $u$  is  $x$ ,  $y$  or  $z$ . Therefore a chaotic oscillation caused by the sequence of period-doubling bifurcations is entirely synchronized, see Fig. 5.

Second, we consider the successive occurrences of



**Fig. 4** Bifurcation diagram. The curve denotes the period-doubling bifurcation. The areas in which stable periodic points  ${}_0D_0^1$ ,  ${}_0D_0^2$ , and  ${}_0D_0^4$  exist are, respectively, indicated by shading [dotted], [cross-hatched], and [solid black].



**Fig. 5** Trajectory corresponding to the periodic point  ${}_0D_0^{2^n}$  and chaos. (Left) Phase portrait. (Right) Waveform. All solutions satisfy  $x_1(t) = x_2(t)$ . Figure 4 shows this cascade of period-doubling bifurcations.

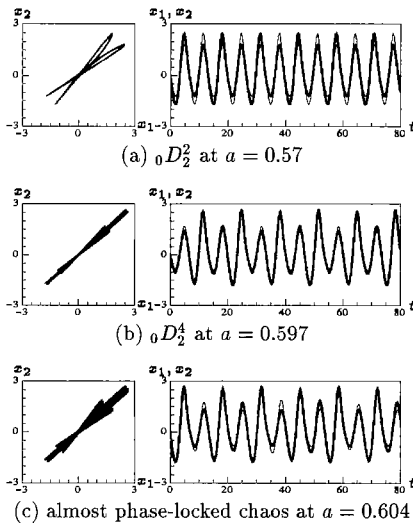
parameter regions in which the stable periodic point  ${}_0D_2^{2^n}$  is generated by a D-type of branching

$${}_1D_0^{2^n} \Rightarrow {}_0D_2^{2^n} + {}_1^*D_2^{2^n} + {}_1^{**}D_2^{2^n} \quad (10)$$

The bifurcated stable periodic point has property on its corresponding waveforms with period  $T$ :

$$u_1(t + T/2) = u_2(t) \quad (11)$$

where  $u$  is  $x$ ,  $y$  or  $z$  of the periodic solution. Because the property of (11) is preserved for a large  $n$ , the sequence causes an almost synchronized chaotic oscillation, see



**Fig. 6** Trajectory corresponding to the periodic point  ${}_0D_2^{2^n}$  and chaos. In the figure for waveform, heavy and light curves denote  $x_1$  and  $x_2$ , respectively.

Fig. 6. The corresponding rotation sequence for  ${}_0D_2^{2^n}$  of (6) is  $[LL \cdots LRR \cdots R]$  with  $2^n$  bits for  $n \geq 1$ .

Third, we show a coexistence of another type of almost synchronized chaotic oscillation, which is generated by a doubling-process of periodic points denoted by  ${}_0D_4^{2^n}$ . The corresponding waveforms with period  $T$  have property such that

$$u_1(t + T/4) = u_2(t) \tag{12}$$

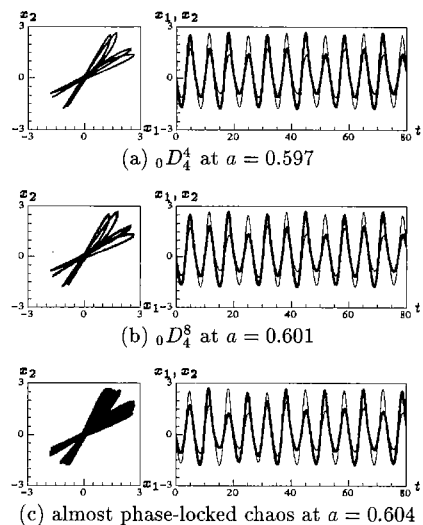
where  $u$  is  $x, y$  or  $z$  of the periodic point  ${}_0D_4^{2^n}$  for a finite number of  $n$ . The corresponding rotation sequence for  ${}_0D_4^{2^n}$  of (6) is  $[LL \cdots LRR \cdots RLL \cdots LRR \cdots R]$  with  $2^n$  bits for  $n \geq 2$ . The phase-locking may be almost preserved for a larger  $n$  and then produces another type of chaotic synchronization, see Fig. 7.

**6. Conclusions**

We have proposed a circuit model for the Rössler equation, and investigated synchronization of chaotic oscillations in the coupled Rössler system. Moreover, we have studied three routes to entirely and almost synchronized chaotic attractors from phase-locked periodic oscillations. It is conjectured that there exist many sequences for producing chaotic oscillations with various phase-locking modes. We also have shown that locking phenomena observed in the coupled Rössler system can be reduced to properties of periodic points in the coupled quadratic map.

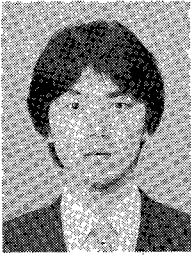
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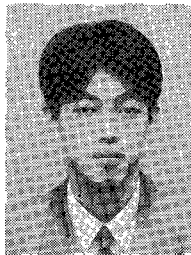
**Fig. 7** Trajectory corresponding to the periodic point  ${}_0D_4^{2^n}$  and chaos. The meaning of line styles in the figure is the same as Fig.6.

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