Bifurcation of periodic states in coupled BVP oscillators with hard characteristics

Hiroyuki Kitajima and Hiroshi Kawakami

The University of Tokushima, Tokushima, 770-8506 Japan, kitaji@ee.tokushima-u.ac.jp

Abstract— In a previous report, we classified periodic solutions into eight different types according to their symmetrical properties. However stable solutions were only two types. In this report, to obtain more stable states, we investigate bifurcations observed in two identical BVP oscillators with fifth order characteristics coupled by an inductor containing resistive component. We consider two cases that the single oscillator has (1) hard oscillation and (2) three stable equilibrium points. In total we obtain six kinds of periodic solutions. Moreover in the case (2), chaotic attractors caused by a cascade of period-doubling bifurcations are found.

I. Introduction

Systems of coupled oscillators with fifth order characteristics have been investigated [1–4]. Datardina et al. studied two coupled oscillators [1] and Endo et al. considered N coupled oscillators in a ladder structure [2]. They obtained that there exist three kinds of stable states: zero (equilibrium point at the origin), two single-modes (in-phase and anti-phase solution) and a double mode (quasi-periodic solution). However by using symmetrical properties of the systems we predict that many single modes not referred in [1, 2] will exist.

In [5] we theoretically classified equilibrium points and periodic solutions into four and eight types, respectively, according to their symmetrical properties. We numerically confirmed the existence of the classified solutions, however stable periodic solutions are inphase and asymmetrical solutions, because the single oscillator has a stable periodic solution invariant under inversion of state variables and equilibrium points not the origin.

One of our aim of this study is to obtain more stable periodic solutions classified in [5]. Therefore we construct the system of two identical oscillators with fifth order characteristics coupled by an inductor containing resistive component. By obtaining bifurcation sets of periodic solutions we clarify parameter regions in which the classified solutions are stably exist and the appearance mechanisms of the classified solutions. Moreover we observe chaotic attractors caused by successive period-doubling bifurcations of asymmetrical periodic solutions.

II. Circuit Equation

We consider an inductively coupled oscillator system shown in Fig. 1. The circuit equations are described as

$$\begin{aligned} L\frac{di_1}{dt} &= v_1 - ri_1 \\ C\frac{dv_1}{dt} &= -i_1 - g(v_1) - i_3 \\ L\frac{di_2}{dt} &= v_2 - ri_2 \\ C\frac{dv_2}{dt} &= -i_2 - g(v_2) + i_3 \\ L_0\frac{di_3}{dt} &= v_1 - v_2 - R_0 i_3 \end{aligned} \tag{1}$$

where nonlinear conductance $g(v)$ has fifth order characteristics:

$$g(v) = a_1 v + a_3 v^3 + a_5 v^5. \tag{2}$$

After normalization we obtain

$$\begin{aligned} \frac{dx_1}{dt} &= \omega y_1 - \sigma x_1 \\ \frac{dy_1}{dt} &= -\alpha y_1 - \beta y_1^3 - \gamma y_1^5 - \omega x_1 - \omega_0 x_3 \\ \frac{dx_2}{dt} &= \omega y_2 - \sigma x_2 \\ \frac{dy_2}{dt} &= -\alpha y_2 - \beta y_2^3 - \gamma y_2^5 - \omega x_2 + \omega_0 x_3 \\ \frac{dx_3}{dt} &= -\sigma_0 x_3 + \omega_0 (y_1 - y_2) \end{aligned} \tag{3}$$

where

$$\begin{aligned} & y_k = \sqrt{C} v_k,\ x_k = \sqrt{L} i_k,\ (k = 1,\ 2),\ x_3 = \sqrt{L_0} i_3, \\ & \omega = \frac{1}{\sqrt{LC}},\ \omega_0 = \frac{1}{\sqrt{L_0 C}},\ \sigma = \frac{r}{L}, \\ & \sigma_0 = \frac{R_0}{L_0},\ \alpha = \frac{a_1}{C},\ \beta = \frac{a_3}{C^2},\ \gamma = \frac{a_5}{C^3}. \end{aligned} \tag{4}$$

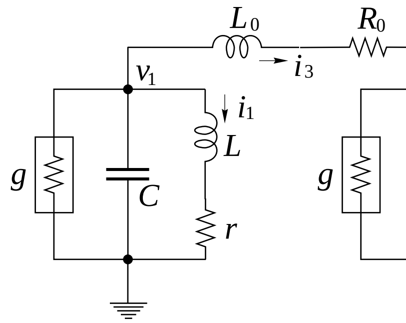


Figure 1: Circuit diagram.

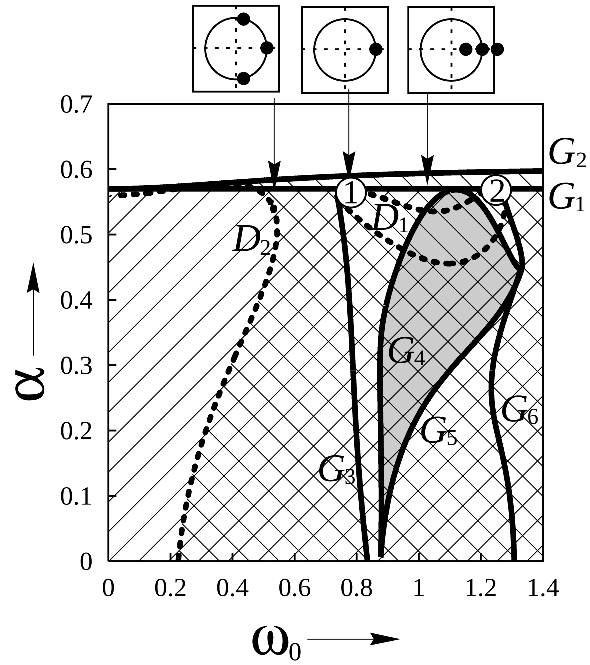


Figure 2: Bifurcation diagram. The subscript 1 and 2 denote the bifurcation sets of the in-phase and the anti-phase solution, respectively. G_3 to G_6 are tangent bifurcation sets of the almost in-phase solution generated by D-type of branching D_1 of the in-phase solution. Small squares above the bifurcation diagram represent the location of characteristic multipliers in a complex plane.

III. Results

We fix the parameters in Eqs. (3) as

$$\beta = -1.4,\ \gamma = 0.4,\ \sigma_0 = 0.5,\ \sigma = 0.5.$$

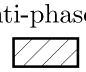
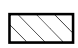
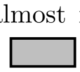
The parameter ω_0 in Eqs. (3) can be considered as coupling coefficient, so we choose α and ω_0 as bifurcation parameters. We study bifurcation problems in two cases:

1. $\omega = 1.0$. The single oscillator has a stable periodic solution and a stable equilibrium point at the origin. This state is called hard oscillation.

2. $\omega = 0.5$. The single oscillator has only stable equilibrium points.

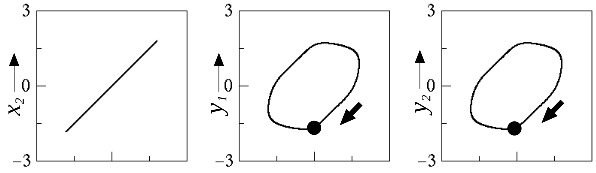
In the following bifurcation diagrams, the symbols G, D, N and I indicate tangent, pitchfork (D-type of branching), Neimark-Sacker and period-doubling bifurcation, respectively.

A. $\omega = 1.0$

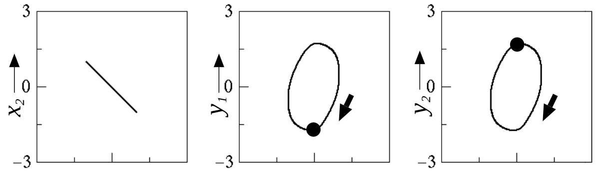
We show a bifurcation diagram in Fig. 2. In this figure we observe stable in-phase, anti-phase and almost in-phase solutions in the region ,  and , respectively. Each solution is shown in Fig. 3. Because the single oscillator has a stable periodic solution, we observe the stable in-phase solution. Moreover the coupling element contains an inductor, the anti-phase solution stably exists. This stable anti-phase solution never exists in a resistively coupled oscillator with voltage ports. The almost in-phase solution is generated by D-type of branching D_1 of the in-phase solution but it is unstable. Through tangent bifurcations G_3 to G_6, the almost in-phase solution becomes stable in the region .

On the tangent bifurcation set G_1 except for between points marked by ① and ② in Fig. 2, the characteristic multipliers satisfy two bifurcation (tangent and Neimark-Sacker) conditions. To clarify this phenomenon we consider the system applied DC term J for the first equation of Eqs. (3). When the parameter ω_0 is less than the point marked by ① (or larger than the point marked by ②), tangent and Neimark-Sacker bifurcation sets intersect on the axis of $J = 0$, see Fig. 4(a). This bifurcation is a degenerate codimension two bifurcation. A quasi-periodic solution generated by the Neimark-Sacker bifurcation N is unstable therefore we can not observe it. At the critical point marked by ① (or ②), the Neimark-Sacker bifurcation set N disappears, see Fig. 4(b), and D-type of branching D_1 corresponding to a cusp point of a tangent bifurcation touches the tangent bifurcation set G_1, see Fig. 2. Between the point marked by ① and ②, the bifurcation structure on (J, α) plane is shown in Fig. 4(c) hence the characteristic multipliers on G_1 satisfy only a tangent bifurcation condition.

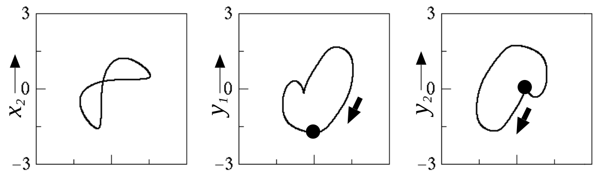
The solutions shown in Fig. 3(c) and Fig. 3(d) meet Neimark-Sacker bifurcations and generate quasi-periodic solutions, see Fig. 5(a) and (b). These quasi-periodic solutions are stably observed in a wide parameter range because the parameter values of ω and ω_0 are different.



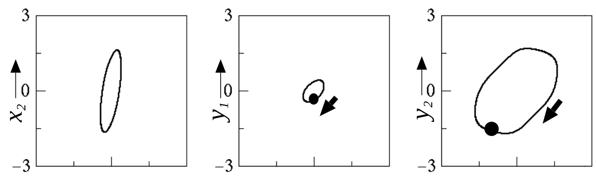
(a) In-phase. $\alpha = 0.4$. $\omega_0 = 1.0$.



(b) Anti-phase. $\alpha = 0.4$. $\omega_0 = 1.0$.



(c) Almost in-phase. $\alpha = 0.4$. $\omega_0 = 1.0$.



(d) $(\bar{I}_5, L/2)$-symmetrical solution [5]. $\alpha = 0.4$. $\omega_0 = 0.45$.

Figure 3: Phase portrait. Arrows and the points marked by closed circles indicate the time direction of the trajectory and the fixed point of Poincaré map, respectively. (Left) Oscillator 1 vs. Oscillator 2. (Middle) Oscillator 1. (Right) Oscillator 2.

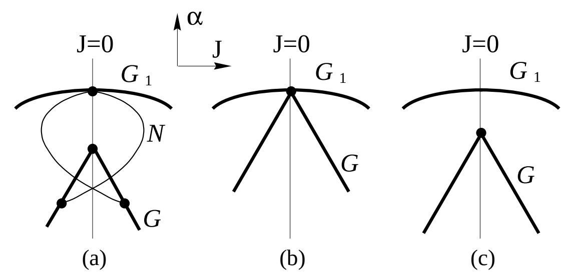
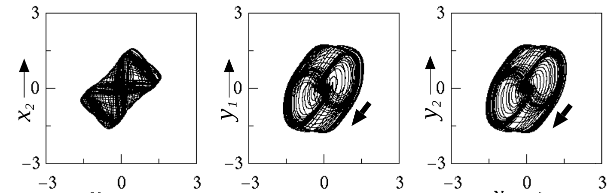
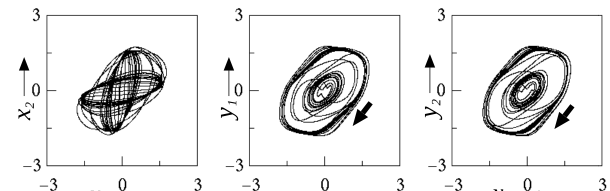


Figure 4: Schematic bifurcation diagram.



(a) Quasi-periodic solution from almost in-phase solution. $\alpha = 0.4$. $\omega_0 = 0.88$.



(b) Quasi-periodic solution from $(\bar{I}_5,\ L/2)$-symmetrical solution. $\alpha = 0.4$. $\omega_0 = 0.5$.

Figure 5: Phase portrait of quasi-periodic solution.

B. $\omega = 0.5$

We observe stable almost anti-phase solutions, see Fig. 6(a), generated by D-type of branching of the anti-phase solution. These almost anti-phase solutions meet D-type of branchings and generate asymmetrical periodic solutions shown in Fig. 6(b). These two solutions: almost anti-phase and asymmetrical solutions can not be confirmed in the system of $\omega = 1.0$. Decreasing the parameter ω_0 asymmetrical solutions meet successive period-doubling bifurcations and generate chaotic attractors, see Fig. 7. One-parameter bifurcation diagram of two asymmetrical solutions is shown in Fig. 8. At $\omega_0 \simeq 0.5355$ two chaotic attractors are merged into one attractor. An interesting point is that even though the single oscillator has only equilibrium points, by changing the value of the coupling inductor we obtain not only an anti-phase periodic solution but also chaotic attractors.

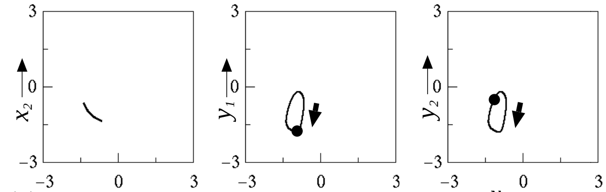
IV. Concluding Remarks

Bifurcations of periodic states observed in a coupled oscillator with hard characteristics have been investigated. We considered two cases that the single oscillator has (1) oscillatory solution and (2) only equilibrium points. We obtained the following things in each case:

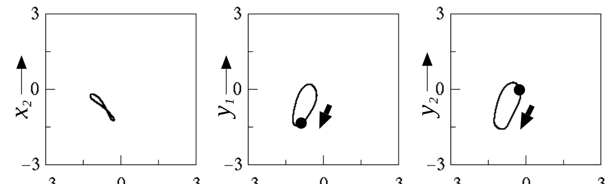
(1) Stable in-phase, anti-phase and almost in-phase solutions are confirmed. We found degenerate codimension two bifurcation sets which satisfy both tangent and Neimark-Sacker bifurcation conditions. Considering the system applied DC term, we clarified the bifurcation structure and obtained that the quasi-periodic solution caused by the Neimark-Sacker bifurcation is unstable.

(2) Even though the single oscillator has only equilibrium points, the coupled system has an anti-phase periodic solution because the system coupled by a state variable. By calculating symmetry-breaking bifurcation set, the anti-phase solution bifurcates to asymmetrical solutions. Moreover we found chaotic attractors caused by a cascade of period-doubling bifurcations of asymmetrical solutions.

To analyze the case of a large number of coupled oscillators is an interesting problem for the future.



(a) Almost anti-phase solution. $\alpha = 0.31$. $\omega_0 = 0.8$. $\omega = 0.5$.



(b) Asymmetrical solution. $\alpha = 0.52$. $\omega_0 = 0.539$. $\omega = 0.5$.

Figure 6: Phase portrait.

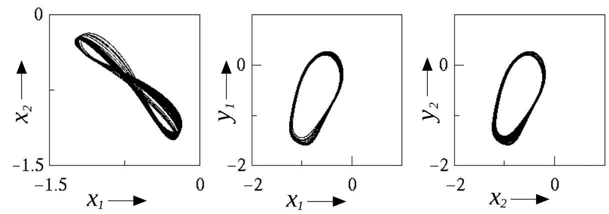


Figure 7: Chaotic attractor. $\alpha = 0.522$. $\omega_0 = 0.5356$. $\omega = 0.5$.

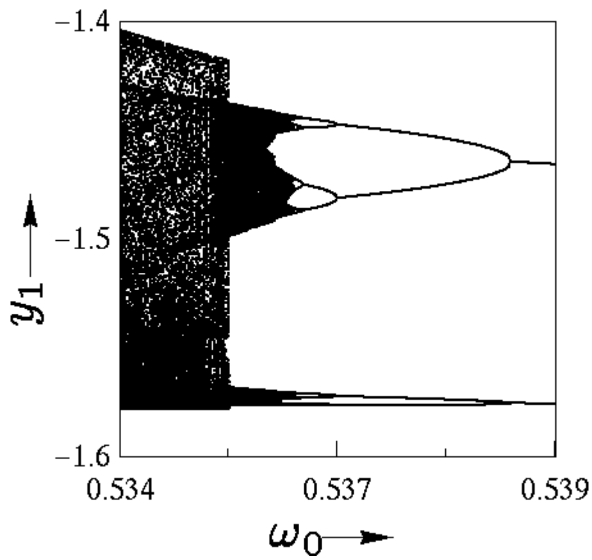


Figure 8: One-parameter bifurcation diagram. $\alpha = 0.522$. $\omega = 0.5$.

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