Bifurcation and chaos in unidirectionally coupled oscillators

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Abstract— Bifurcations of equilibrium points and limit cycles in unidirectionally coupled three oscillators are studied. According to their symmetrical properties, we classify equilibrium points and limit cycles into three and eight different types, respectively. Possible oscillations in unidirectional coupled three oscillators are presented by calculating Hopf bifurcation sets of equilibrium points. We also observe chaotic oscillation caused by a cascade of period-doubling bifurcations.

I. Introduction

Systems of coupled oscillators are good models for biological rhythmic oscillations such as human circadian rhythms [1], finger movements [2], animal locomotion [3] and so on. The investigators have studied the mechanism of oscillation and phase transitions between distinct oscillatory modes. From the standpoint of bifurcation, the former and the latter correspond to Hopf bifurcation of an equilibrium point and D-type of branching of a periodic solution, respectively. Using group theory, it has been possible to derive some general theorems concerning with the bifurcations of equilibrium points.

In Ref. [4], we studied bifurcations of equilibrium points and periodic solutions observed in coupled two oscillators with voltage ports. We classified equilibrium points and periodic solutions into four and eight different types, respectively, according to their symmetrical properties. By calculating D-type of branching sets (symmetry-breaking bifurcations) of equilibrium points and periodic solutions, we showed that all types of equilibrium points and periodic solutions are systematically found.

In this report we consider a ring BVP oscillator network coupled unidirectionally by linear resistor. This connection is interesting for studying the bifurcational mechanisms of coupled oscillator. Especially to explain the appearance of phase shift oscillations uniquely arisen by coupling effect the model serves as a prototype circuit. We classify the equilibrium points and the periodic solutions into three and eight different types, respectively, according to their symmetrical properties. Possible oscillations in three unidirectionally coupled oscillators are presented by calculating Hopf bifurcation sets of equilibrium points. By calculating bifurcation sets of periodic solutions, transitions between the solutions with different symmetrical properties are discussed. Moreover we observe chaotic repellor without symmetry created by a cascade of period-doubling bifurcations.

II. Circuit Equation and Related Property

We assume nonlinear conductance $g(v)$ in Fig. 1 as

$$g(v) = -a_1 v + a_3 v^3. \tag{1}$$

By connecting the output port b-b' to the next input port a-a' consecutively as a ring we realize a coupled oscillator circuit [5]. In this report we consider three oscillators case thus the normalized circuit equations are described by

$$\begin{aligned} \frac{dx_k}{dt} &= \omega y_k - \sigma x_k \\ \frac{dy_k}{dt} &= -\omega x_k + \epsilon(1-\beta y_k^2) y_k - \delta(y_k + y_{k-1}) \\ & \qquad (k = 1, \cdots, 3, \quad y_0 \equiv y_3) \end{aligned} \tag{2}$$

where

$$\begin{aligned} &\omega = \frac{1}{\sqrt{LC}},\ \epsilon = \frac{a_1}{C},\ \beta = \frac{a_3}{a_1 C},\ \sigma = \frac{r}{L}, \\ &\delta = \frac{G}{C},\ x_k = \sqrt{C} v_k,\ y_k = \sqrt{L} i_k. \end{aligned} \tag{3}$$

We define matrices as:

$$P = \begin{bmatrix} O & I_2 & O \\ O & O & I_2 \\ I_2 & O & O \end{bmatrix}, \hat{I}_6 = -I_6 \tag{4}$$

where I_n is $n \times n$ identity matrix and O is 2×2 zero matrix. We also define matrix group:

$$\Gamma = \{I_6, P, P^2, \hat{I}_6, \hat{I}_6 P, \hat{I}_6 P^2\}. \tag{5}$$

Then Eqs. (2) is equivariant under the coordinate transformation:

$$\boldsymbol{x} \mapsto G\boldsymbol{x};\ \boldsymbol{y} \mapsto G\boldsymbol{y},\ \forall G \in \Gamma \tag{6}$$

Note that Γ has a cyclic and an inversion subgroups:

$$C_3 = \{I_6, P, P^2\},\ I = \{I_6, \hat{I}_6\} \tag{7}$$

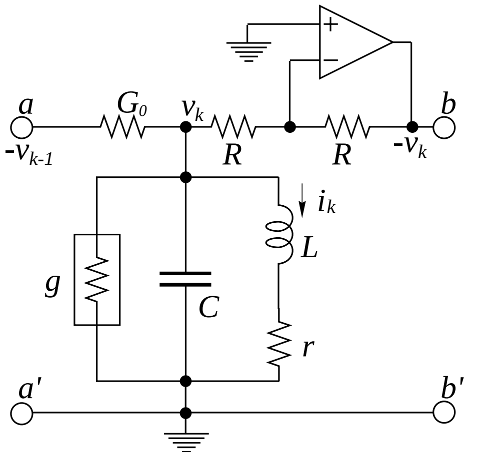


Figure 1: Circuit diagram.

III. Results

We fix the parameters in Eqs. (2) as

$$c_3 = 1/3,\ \beta = 1.0,\ \omega = 1.0,\ \sigma = 0.5.$$

For these parameters the single oscillator has limit cycles thus coupled oscillators have in-phase solutions.

A. Classification of equilibrium points and limit cycles

To clarify the symmetrical property of trajectories we use the following coordinate transformation:

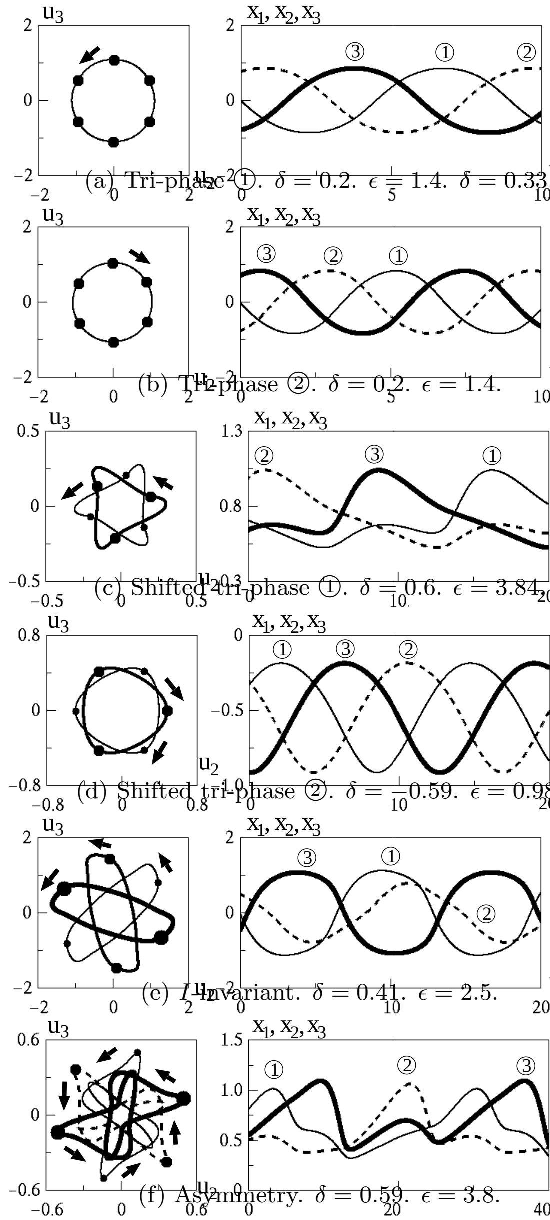
$$\boldsymbol{u} = Q\boldsymbol{x},\ \boldsymbol{v} = Q\boldsymbol{y} \tag{8}$$

where

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \tag{9}$$

We classify equilibrium points into three different types according to their symmetrical properties: (1) the origin, (2) C_3-invariant and (3) asymmetry. Type (1) is invariant under transformation (6). On the other hand type (3) has no symmetrical operations. Therefore if we can find one equilibrium point of type (3), then six equilibrium points exist by the coordinate transformation (6).

We also classify limit cycles into eight different types according to their symmetrical properties. Figure 2 shows six kinds of limit cycles. Six points marked by closed circles in each figure indicate the coordinate transformation by the elements in Γ. Tri-phase solutions ((a)–(d)) are invariant under transformation by the elements in C_3 with time shift $L/3$ where L is the period of the limit cycle. I-invariant solution (e) is invariant under transformation by the element in I with time shift $L/2$. In addition to Fig. 2 there exist in-phase and "shifted in-phase" limit cycles. Hence in total eight different limit cycles exist in Eqs. (2).



(a) Tri-phase ①. $\delta = 0.2$. $\epsilon = 1.4$. $\delta = 0.33$.

(b) Tri-phase ②. $\delta = 0.2$. $\epsilon = 1.4$.

(c) Shifted tri-phase ①. $\delta = 0.6$. $\epsilon = 3.84$.

(d) Shifted tri-phase ②. $\delta = 0.59$. $\epsilon = 0.98$.

(e) I-invariant. $\delta = 0.41$. $\epsilon = 2.5$.

(f) Asymmetry. $\delta = 0.59$. $\epsilon = 3.8$.

Figure 2: Classification of periodic solutions according to their symmetrical properties. Arrows indicate the time direction of trajectory. (Left) Phase portrait. (Right) Waveform.

B. Bifurcation diagrams

We show in Fig. 3 a bifurcation diagram of equilibrium points type (1) and (2). In the region ▒ stable equilibrium point of type (1) exists. Table 1 shows that each bifurcation set in Fig. 3 generates what kinds of equilibrium points and limit cycles. The eigenspace of symmetrical equilibrium points of type (1) and (2) is 2-dimensional in-phase direction and 4-dimensional tri-phase direction. Thus the direction of their bifurcation is restricted to in-phase or tri-phase direction. Therefore I-invariant limit cycles never occur by Hopf bifurcation of the equilibrium point.

In Fig. 3 open circle represents the point of intersection of double Hopf bifurcations, called Hopf-Hopf bifurcation in [6]. From this point two Neimark-Sacker bifurcations appear and generate quasi-periodic solution [6]. Square in Fig. 3 represents the point of intersection of D-type of branching and Hopf bifurcation. This codimension three bifurcation is not refereed in [6, 7] therefore we study bifurcation structure around it.

Figure 4 shows a bifurcation diagram around the point of intersection of ${}_0d_1$ and ${}_0h_3$ (Subscript 0 indicates the bifurcation of the origin). From this point, D-type of branching and Neimark-Sacker bifurcation of limit cycle and Hopf bifurcation of equilibrium point necessarily appear. The mechanisms are as follows: Tri-phase ② solution (Fig. 2(b)) caused by ${}_0h_3$ meets D-type of branching D (symmetry-breaking bifurcation) and generates "shifted tri-phase ②" solution (Fig. 2 (d)). The "shifted tri-phase ②" solution crossed Neimark-Sacker bifurcation disappears by Hopf bifurcation ${}_1h_3$ of C_3-invariant equilibrium point generated by ${}_0d_1$.

Finally we show a bifurcation diagram of equilibrium points without symmetry in Fig. 5. Closed circle denotes the point of intersection of tangent and Hopf bifurcation called Fold-Hopf bifurcation in [6]. In the region [light grey box] and [dark grey box] there exist completely unstable and stable equilibrium points without symmetry, respectively. By crossing the Hopf bifurcation set h from the region [light grey box], completely unstable limit cycle is generated. Increasing the parameter ϵ the limit cycle meets a cascade of period-doubling bifurcations and become chaotic repellor, see Fig. 6.

Table 1: Bifurcations in Fig. 3

notation	bifurcation
${}_0d_1$	the origin ⇔ C_3-invariant
${}_0h_1$	the origin ⇔ in-phase
${}_0h_2$	the origin ⇔ tri-phase ①
${}_0h_3$	the origin ⇔ tri-phase ②
${}_1h_1$	C_3-invariant ⇔ shifted in-phase
${}_1h_2$	C_3-invariant ⇔ shifted tri-phase ①
${}_1h_3$	C_3-invariant ⇔ shifted tri-phase ②

IV. Concluding Remarks

We consider a system of coupled unidirectionally three BVP oscillators. Equilibrium points and limit cycles were classified into three and eight different types,

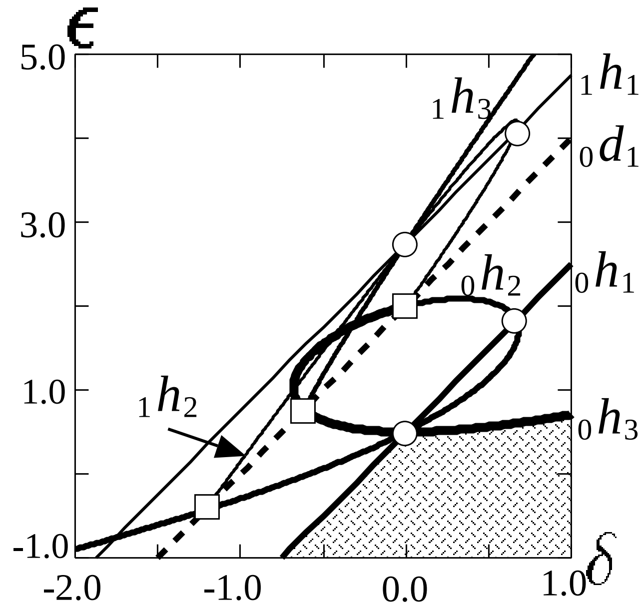


Figure 3: Bifurcation diagram of symmetrical equilibrium points. The symbols d and h indicate D-type of branching and Hopf bifurcation set, respectively.

respectively, according to their symmetrical properties. By calculating Hopf bifurcation sets of all types of equilibrium points, possible oscillations in the system were shown. We obtained bifurcation diagrams of limit cycles and clarified the bifurcation structure of the point of intersection of D-type of branching and Hopf bifurcation called codimension three bifurcation. Chaotic oscillations without symmetry created by a cascade of period-doubling bifurcations were found.

To analyze the case of a large number of coupled oscillators is an interesting problem for the future.

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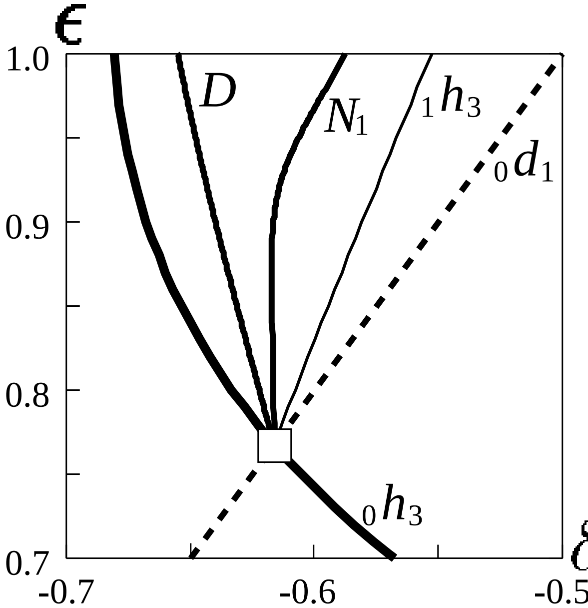


Figure 4: Bifurcation diagram. The symbols N and D indicate Neimark-Sacker bifurcation and D-type of branching set of limit cycle, respectively.

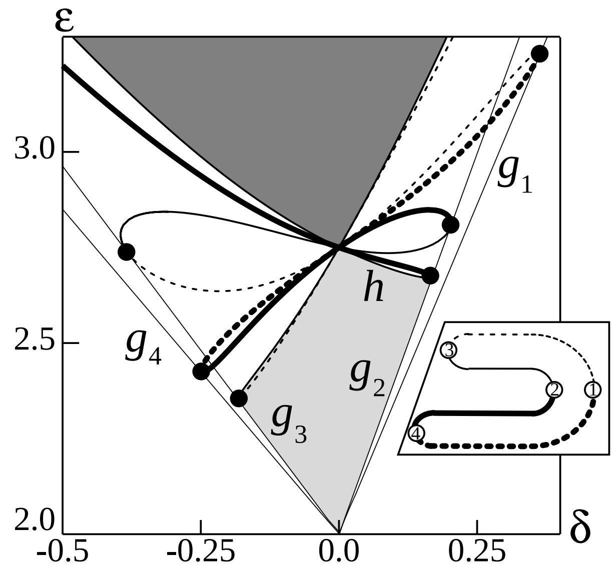
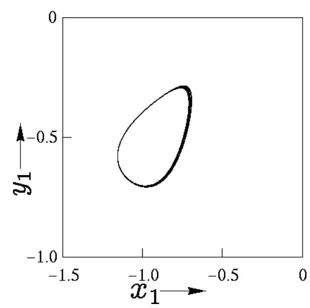
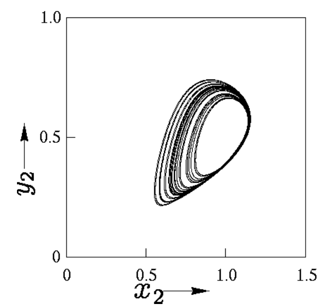


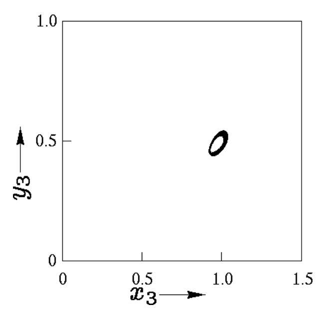
Figure 5: Bifurcation diagram of equilibrium points without symmetry. The symbol g indicates tangent bifurcation and another curves represent Hopf bifurcation. In a trapezoid manifolds of equilibrium points are shown. ① to ④ indicate tangent bifurcation corresponding to g_1 to g_4, respectively.



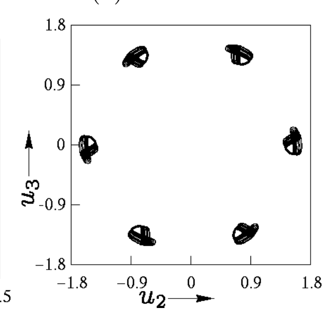
(a) Oscillator 1



(b) Oscillator 2



(c) Oscillator 3



(d) Six chaotic repellor by transformation (6).

Figure 6: Chaotic oscillation without symmetry. $\delta = 0.06$. $\epsilon = 2.81$.

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