Synchronization in Two Coupled Oscillators with Three Ports

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1 Introduction

In this paper, we investigate the bifurcations of periodic solutions observed in coupled Modified Bonhöffer-van der Pol (MBVP) equations. Each oscillatory circuit is shown Fig. 1 and is described by

$$\begin{aligned} L_{k1}\frac{di_{k1}}{dt} &= v_k - E_{k1} \\ L_{k2}\frac{di_{k2}}{dt} &= v_k - E_{k2} \\ C_k\frac{dv_k}{dt} &= -g_k(v_k) - i_{k1} - i_{k2} \end{aligned} \qquad (1)$$

where the nonlinear conductance g_k is defined by

$$g(v) = -v + v^3/3$$

In the MBVP equation, we observe various phenomena such as oscillations with alternately appearance of large and small amplitudes, many period-doubling cascades of limit cycles, chaotic oscillations caused by the infinite doubling process, and so on. This system is considered as an oscillator with three ports from which we can observe the states of the oscillator: the state variables i_{k1}, i_{k2} and v_k are respectively observable fromI_1-port, I_2-port, and V-port in Fig. 1. Combining these two identical oscillators through three ports by a resistor, we can realize six kinds of oscillator networks: these types of connections are symbolically denoted by I_1-I_1, I_2-I_2, I_1-I_2, I_1-V,I_2-V and V-V.

Now we only consider V-V type connection, so the system equations are given by

$$\begin{aligned} L_1\frac{di_{11}}{dt} &= v_1 - E_1 - R_1 i_{11} \\ L_2\frac{di_{12}}{dt} &= v_1 - E_2 - R_2 i_{12} \\ C\frac{dv_1}{dt} &= -g(v_1) - i_{11} - i_{12} - G_0(v_1 - v_2) \\ L_1\frac{di_{21}}{dt} &= v_2 - E_1 - R_1 i_{21} \\ L_2\frac{di_{22}}{dt} &= v_2 - E_2 - R_2 i_{22} \\ C\frac{dv_2}{dt} &= -g(v_2) - i_{21} - i_{22} - G_0(v_2 - v_1) \end{aligned} \qquad (2)$$

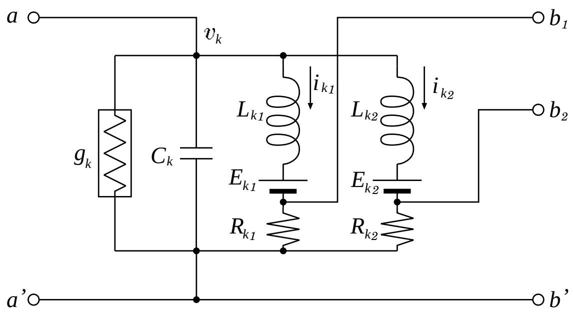


Figure 1: A modified BVP oscillator with three ports: V-port(terminals a and a'),I_1-port(terminals b_1 and b') and I_2-port (terminals b_2 and b').

2 Method of analysis

Equations (2) are rewritten as:

$$\dot{x} = f(x, \lambda) \tag{3}$$

We assume that Eqs.(2) has a periodic solution with initial condition $x^0 := x(0)$, denoted by $x(t) = \psi(t, 0, x^0)$ with $x(0) = \psi(0, 0, x^0) = x^0$. We define a Poincaré section Π for the trajectory $\psi(t, 0, x^0)$, then the Poincaré mapping T_λ is defined as

$$T_\lambda : \Pi \to \Pi; x \longmapsto \psi(\tau, 0, x) \tag{4}$$

where τ is the time taken for the path of trajectory, which starts from x and ends at firstly return point to Π.

If a solution $\psi(t, 0, x^0)$ is a periodic, then the point x^0 is a fixed point of T_λ:

$$F_\lambda(x^0) := x^0 - T_\lambda(x^0) = 0 \tag{5}$$

Computation of a periodic solution of Eqs. (2) has now been reduced to find $x^0 \in R^6$ satisfying Eq. (5).

We have three types of generic bifurcations (tangent, period-doubling, Hopf bifurcations) and a degenerate bifurcation (D-type of branching) for the fixed point. D-type of branching may appear in the system which possess a symmetrical property. This type of bifurcation occurs when a real eigenvalue passes through the point (1,0) in complex plane. Then the bifurcation condition is a degenerate case of the tangent bifurcation. These bifurcation phenomena can be traced out by solving the fixed point equation and bifurcation condition simultaneously [3].

We use the notation ${}_kD^m$ (resp. ${}_kI^m$) denoting a type having even (resp. odd) number of characteristic multipliers on the real axis $(-\infty, -1)$, k indicates the number of characteristic multiplier outside the unit circle in the complex plane, and m indicates m periodic point of T_λ. If $m = 1$, it will be omitted.

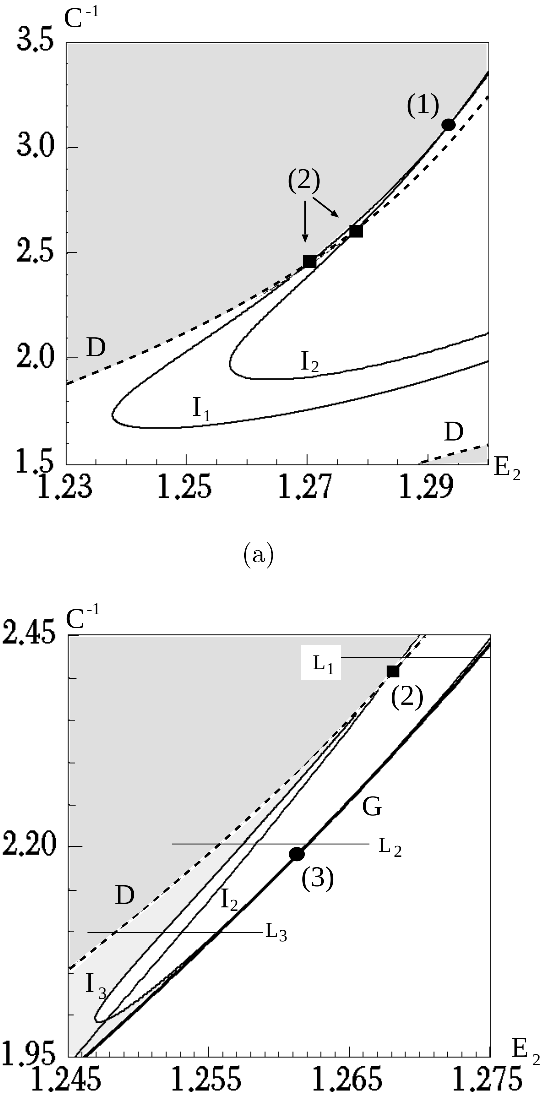
3 Results

Now we show some numerical results of bifurcation diagrams and behavior of the solutions observed in Eqs. (2). In the following we fix several parameters in Eqs. (2) as

$$R_1 = R_2 = 0.5, \; E_1 = 0.2$$

$$L_1^{-1} = 0.4, \; L_2^{-1} = 0.1, \; G_0 = 0.01$$

and consider parameter plane for C^{-1} and E_2.



(a)

(b)

Figure 2: Bifurcation diagram of fixed point of T_λ for Eqs. (2). Stable fixed points are found in shaded portions.

Figure 2(a) and (b) show the bifurcation diagrams of T_λ for Eqs. (2). In these figures dotted line D, heavy line G, and light line I indicate D-type of branching, tangent and period-doubling bifurcations, respectively. In Fig. 2 we observe several codimension-two bifurcations:

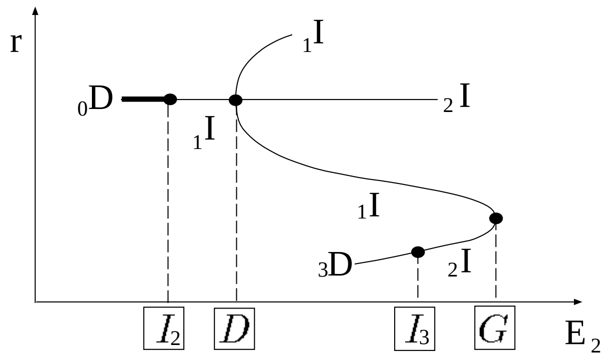
(1) Intersection of period-doubling bifurcations (called double period-doubling bifurcation).

(2) Intersection of period-doubling bifurcation and D-type of branching.

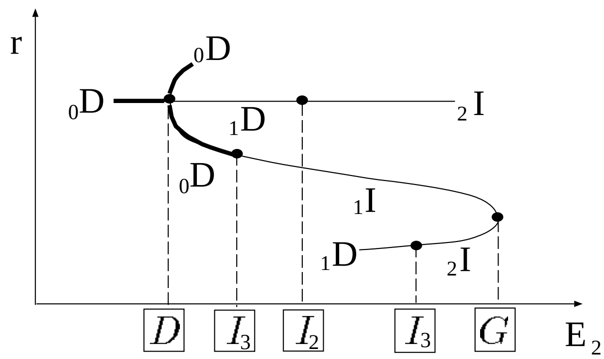
(3) Intersection of tangent and period-doubling bifurcation.

Figure 3 shows a schematically illustrated one-parameter bifurcation diagram where the abscissa is E_2 and the ordinate is a norm of state variables for the fixed point. In the figure, the heavy and light curves, respectively, represent stable and unstable fixed points, and the circled point indicates a bifurcation whose type is indicated at the bottom of the figure.

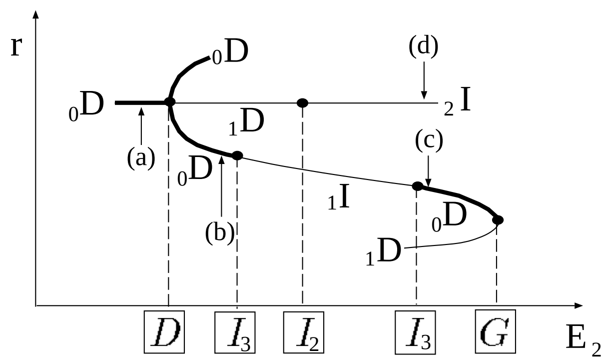
On the left half of line L_2 in Fig. 2(b) there are D-type branching, I_2 and I_3 bifurcations, see Fig. 3(b). From the line L_2 as C^{-1} increases, D-type of branching, I_2 and I_3 bifurcations get closely and they intersect at point (2) of Fig. 2(b). Then D-type of branching and I_2 bifurcation of original fixed point exchange the order and I_3 bifurcation of fixed point generated by D-type of branching disappears, see Fig. 3(a). Because of exchanging order of D-type of branching and I_2 bifurcation, *stable* 2-period solution (resp. fixed point) appears in Fig. 3(a) (resp. Fig. 3(b)).



(a)



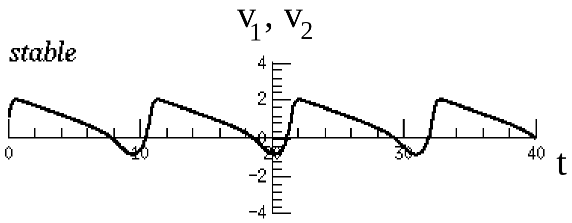
(b)



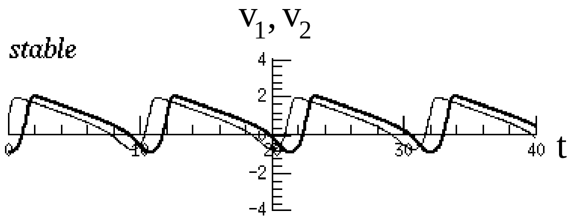
(c)

Figure 3: A schematic diagram for change of the characteristics when E_2 varies along line (a) L_1 ($C^{-1} = 2.43$), (b) L_2 ($C^{-1} = 2.20$) and (c) L_3 ($C^{-1} = 2.10$) in Fig. 2(b) (ordinate r indicates a norm of fixed point).

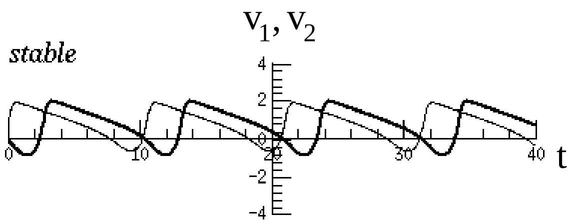
We now discuss the synchronization observed in Fig. 3(c). Figure 4 shows the waveforms corresponding to the parameter points (a)–(d) in Fig. 3(c). At the point (a) v_1 and v_2 are in-phase and stable. After D-type of branching, phase differences appear (b) and increase gradually as E_2 increases (c), then the fixed points disappear by tangent bifurcation. Note that v_1 and v_2 are in-phase but unstable at the point (d).



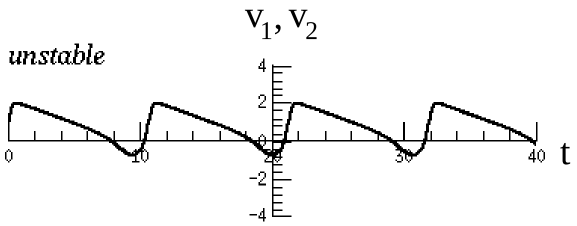
(a)



(b)



(c)



(d)

Figure 4: Waveforms of v_1 and v_2 of Eqs.(2) where $E_2 =$ (a) 1.2484, (b) 1.2518, (c) 1.2558 and (d) 1.2558.

4 Concluding Remarks

We have investigated global properties of bifurcation sets of periodic solutions observed in resistively coupled MBVP equation. We obtain the parameter region in which the in-phase limit cycle is stable. We found new type of the codimension-two bifurcation: double period-doubling bifurcation. The analysis of the bifurcated doubling solutions near this bifurcation is a future problem.

References

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