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BIFURCATION AND CHAOS OF SYNCHRONIZED STATES IN OSCILLATORS WITH HARD CHARACTERISTICS AND STATE COUPLING

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We investigate the phase transition between solutions with distinct symmetrical property observed in a system of coupled three-oscillators with hard characteristics and state coupling. By a symmetry-breaking bifurcation, a symmetrical in-phase solution bifurcates into synchronized modes with a partially in-phase solution and an almost in-phase solution. Moreover, by using the definition of symmetrical and asymmetrical three-phase solutions, we confirmed the existence of a stable symmetrical three-phase quasi-periodic solution and an asymmetrical three-phase chaotic solution in the coupled system.

Keywords: Coupled oscillator; bifurcation; symmetry; three-phase solution.

1. Introduction

Systems of coupled oscillators are widely used as models for biological rhythmic oscillations such as human circadian rhythms [Kronauer *et al.*, 1982; Brown *et al.*, 2000], finger movements [Hirao *et al.*, 1996], animal locomotion [Collins & Stewart, 1993; Golubitsky & Stewart, 1999; Buono & Golubitsky, 2001], swarms of fireflies that flash in synchrony [Winfree, 1980; Kousaka *et al.*, 1998], synchronous firing of cardiac pacemaker cells [Winfree, 1980; Sousa *et al.*, 1994], neural networks [Skinner *et al.*, 1994; Han *et al.*, 1997; Medvedev & Kopell, 2000], and so on.

Using these coupled oscillator models, many investigators have studied the mechanism of generation of synchronous oscillation and phase transitions between distinct oscillatory modes. From the standpoint of bifurcation, the former and the latter correspond respectively to the Hopf bifurcation of an equilibrium point (or the tangent bifurcation of a periodic solution) and to the pitchfork bifurcation (or the period-doubling bifurcation) of a periodic solution. Using a group theoretic discussion applied to the coupled oscillators, we can derive some general theorems concerning the bifurcations of equilibrium points and periodic solutions [Golubitsky *et al.*, 1988].

Our research group has investigated a system of a small number of coupled oscillators, aiming to classify periodic solutions according to their symmetrical properties, and to clarify the phase transition between classified periodic solutions [Papy & Kawakami, 1995a, 1995b; Kitajima & Kawakami, 1998]. We consider that the case of a small number of oscillators is a prototype modeling for understanding the phenomena in the case of a large number of oscillators. Shiohama and Kawakami [1998] studied a system of coupled three oscillators through inductors in a ring. The ring structure is one of the simplest cases of coupled oscillators [Friesen & Stent, 1977; Tsutsumi & Matsumoto, 1984; Ermentrout, 1985], and "three" is the simplest case of the ring structure. They confirmed four kinds of stable periodic solutions and also showed chaotic oscillations caused by successive period-doubling bifurcations.

We have also studied a system of coupled three-oscillators with hard characteristics and state coupling, to obtain more stable states. The single oscillator, called a hard oscillator, has a stable equilibrium point and a stable periodic solution. We have classified the periodic solutions into twelve kinds according to their symmetrical property, and confirmed nine kinds of stable periodic solutions [Yamakawa *et al.*, 1999].

In this paper, we further investigate the phase transition between solutions with distinct symmetrical properties. Moreover, by using the definition of three-phase solutions, we confirmed the existence of a stable symmetrical three-phase quasi-periodic solution and an asymmetrical three-phase chaotic solution. To the best of our knowledge, there is no paper describing an n-phase ($n \geq 3$) chaotic solution based on the common mathematical definition. We consider that these results give useful information for the design of a coupled oscillator system.

2. Preliminaries

2.1. *System equation*

Consider a system of coupled three-oscillators in a ring, as shown in Fig. 1. After normalization of the state variables and parameters, we obtain the following circuit equations:

$$\begin{aligned}\frac{dx_i}{dt} &= -\alpha x_i - \beta x_i^3 - \gamma x_i^5 + \omega x_{i+1} \\ &\quad + \omega_0(x_{i+5} - x_{i+8}) \\ \frac{dx_{i+1}}{dt} &= -\omega x_i - \sigma x_{i+1} \\ \frac{dx_{i+2}}{dt} &= -\sigma_0 x_{i+2} + \omega_0(x_{i+3} - x_{i+6}) \\ &\quad (i = 1, 4, 7, x_{10} \equiv x_1, x_{12} \equiv x_3, x_{13} \equiv x_4, \\ &\quad x_{15} \equiv x_6),\end{aligned} \tag{1}$$

where

$$\begin{aligned} x_1 &= \sqrt{C}v_1,\ x_2 = \sqrt{L}i_1,\ x_3 = \sqrt{L_0}i_4\,, \\ x_4 &= \sqrt{C}v_2,\ x_5 = \sqrt{L}i_2,\ x_6 = \sqrt{L_0}i_5\,, \\ x_7 &= \sqrt{C}v_3,\ x_8 = \sqrt{L}i_3,\ x_9 = \sqrt{L_0}i_6\,, \end{aligned} \tag{2}$$

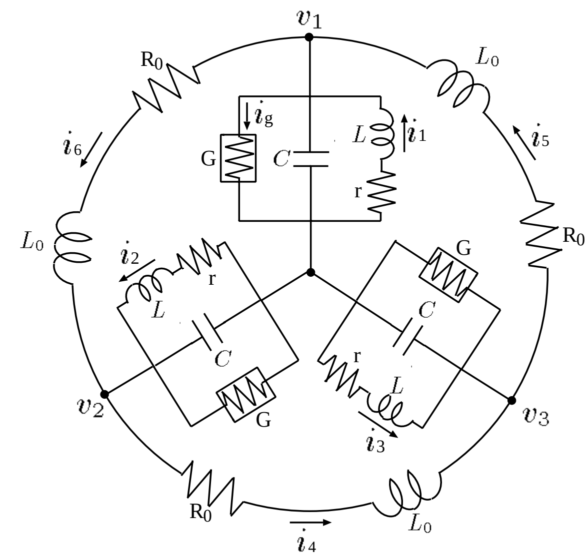

Fig. 1. Circuit diagram. The characteristics of the boxed nonlinear conductance are assumed as $i_g(v_i) = a_1v_i + a_3v_i^3 + a_5v_i^5$.

$$\alpha = \frac{a_1}{C},\ \beta = \frac{a_3}{C^2}, \gamma = \frac{a_5}{C^3},\ \sigma = \frac{r}{L},\ \sigma_0 = \frac{R_0}{L_0}\,,$$
$$\omega = \frac{1}{\sqrt{CL}},\ \omega_0 = \frac{1}{\sqrt{CL_0}}\,.$$

We define the permutation matrix P, the flip matrix Q, and the inversion matrix $\bar{I}_9$, as follows:

$$\begin{aligned} P &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \otimes I_3\,, \\ Q &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \bar{I}_9 = -I_9\,, \end{aligned} \tag{3}$$

where I_n denotes the $n \times n$ identity matrix. We also define a matrix group:

$$\begin{aligned}\Gamma = \{&I_9, P, P^2, Q, QP, QP^2, \bar{I}_9, \bar{I}_9P, \bar{I}_9P^2, \\ &\bar{I}_9Q, \bar{I}_9QP, \bar{I}_9QP^2\}.\end{aligned} \tag{4}$$

Then, Eq. (1) is equivariant to a new system under the following coordinate transform:

$$x \longmapsto gx, \forall g \in \Gamma\,. \tag{5}$$

Note that Γ has a dihedral subgroup:

$$D_3 = \{I_9,\ P,\ P^2,\ Q,\ QP,\ QP^2\}, \tag{6}$$

and D_3 has a cyclic subgroup:

$$C_3 = \{I_9, P, P^2\} \tag{7}$$

together with three conjugate subgroups:

$$C_2 = \{I_9, Q\}, \{I_9, QP\}, \{I_9, QP^2\}. \tag{8}$$

2.2. *Poincaré mapping*

We assume that the solution of Eq. (1) is

$$x(t) = \varphi(t, x_0, \lambda), \tag{9}$$

where x_0 is an initial state:

$$\varphi(0, x_0, \lambda) = x_0, \tag{10}$$

and λ is a parameter. We define a Poincaré section Π for the trajectory $\varphi(t, x_0, \lambda)$. Then, the Poincaré mapping T_λ is

$$T_\lambda : \Pi \longrightarrow \Pi;\ x_0 \longmapsto \varphi(\tau, x_0, \lambda), \tag{11}$$

where τ is the time instant taken for the path of trajectory, which starts from x_0 and ends at the first return point to Π.

2.3. *Definition of symmetrical and asymmetrical three-phase solutions*

We define a mapping T_P as

$$T_P : \Pi \longrightarrow \Pi;\ x_0 \longmapsto P^{-1}\varphi(\tau, x_0, \lambda) \tag{12}$$

and a set $\Sigma(x_0)$ as

$$\Sigma(x_0) = \{T_P^k(x_0) \,|\, k \in N\}. \tag{13}$$

If the set $\Sigma(x_0)$ is invariant under the mapping T_P:

$$T_P(\Sigma(x_0)) = \Sigma(x_0) \tag{14}$$

and is connected, then the solution $\varphi(t, x_0, \lambda)$ is called an asymmetrical three-phase solution [Fiedler, 1988; Katsuta, 1995]. When the matrix $P\bar{I}_9$ is used instead of P, the solution is called a symmetrical three-phase solution, where "symmetrical" indicates that it is invariant under the inversion of state variables.

Definitions of other symmetrical solutions, observed in a system with the dihedral group D_3, can be found from [Katsuta, 1995; Shiohama & Kawakami 1998; Yamakawa *et al.*, 1999].

3. Main Results

We fix the parameter values of Eq. (1) as

$$\begin{aligned} \beta &= -1.4, \quad \gamma = 0.4, \\ \sigma &= 0.5, \quad \sigma_0 = 0.5. \end{aligned} \tag{15}$$

In the bifurcation diagrams shown in this section, the tangent, period-doubling, Neimark–Sacker bifurcation, and D-type of branching (pitchfork bifurcation) sets of an m-periodic solution, are indicated respectively by symbols G_j^m, I_j^m, N_j^m and D_j^m, where j denotes the number that distinguishes different bifurcation sets of the same period. If $m = 1$, it will be omitted.

3.1. *Transition between distinct oscillatory modes*

Figure 2 shows a bifurcation diagram of periodic solutions on the parameter plane (ω_0, α). In the region shaded by and , respectively, a stable symmetrical in-phase solution and a symmetrical partially anti-phase solution exist.

By the degenerate symmetry-breaking pitchfork bifurcation (satisfying double pitchfork bifurcation conditions) D_1, the symmetrical in-phase solution bifurcates into a partially in-phase

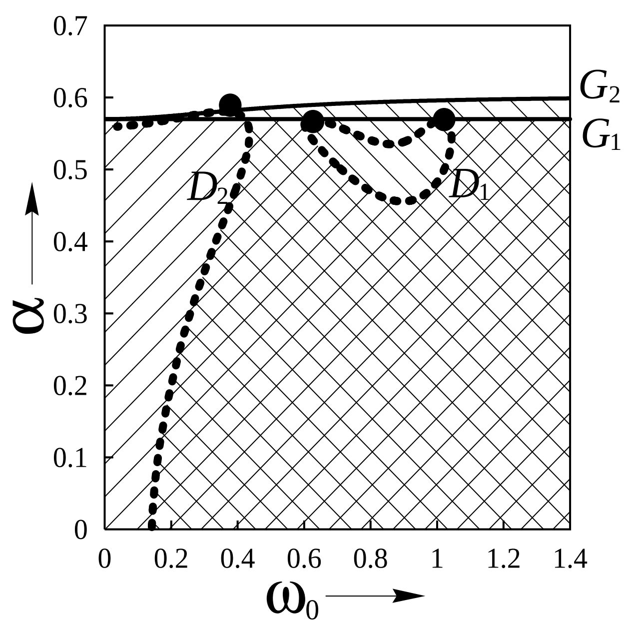

Fig. 2. Bifurcation diagram observed in Eq. (1) with $\omega = 1.0$. The subscripts 1 and 2 of D denote the bifurcation sets of the symmetrical in-phase and the symmetrical partially anti-phase periodic solution, respectively.

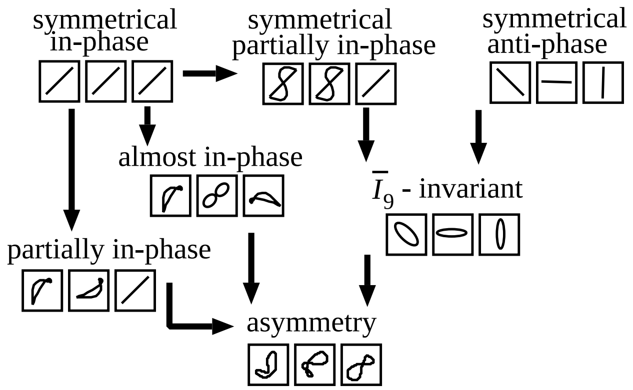

Fig. 3. Transition between solutions with distinct symmetrical property. Three small squares under the name of the solution indicate the trajectory of each solution; (left) x_1 versus x_4, (middle) x_4 versus x_7, (right) x_7 versus x_1.

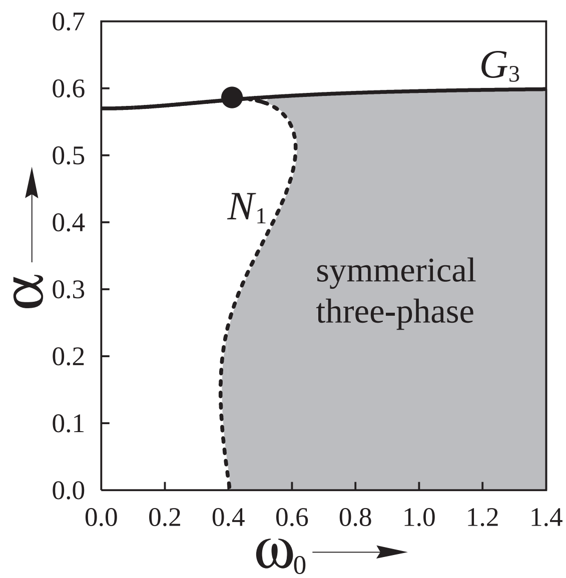

Fig. 4. Bifurcation diagram of symmetrical three-phase periodic solutions observed in Eq. (1) with $\omega = 1.0$.

solution and an almost in-phase solution. By the pitchfork bifurcation D_2 of the symmetrical partially anti-phase solution, $\bar{I}_9$-invariant solution (symmetry with respect to the inversion of state variables) appears.

A summary of the phase transition between solutions with distinct symmetrical property obtained in this paper and those obtained in [Yamakawa *et al.*, 1999] is shown in Fig. 3.

3.2. *Symmetrical three-phase quasi-periodic solution*

A bifurcation diagram of a symmetrical three-phase periodic solution is shown in Fig. 4. In the shaded

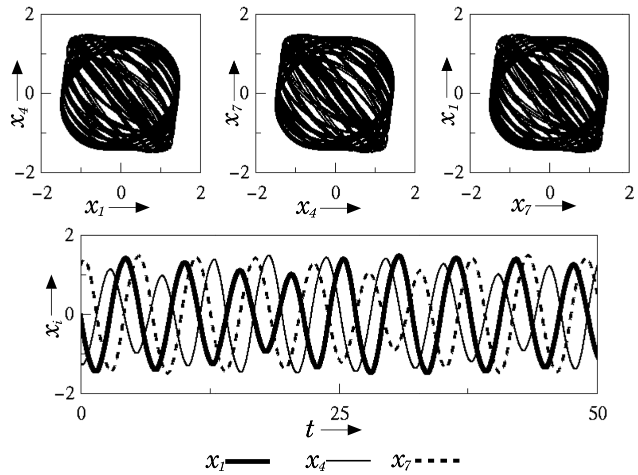

Fig. 5. Symmetrical three-phase quasi-periodic solution: $\alpha = 0.4$, $\omega_0 = 0.5$, $\omega = 1.0$.

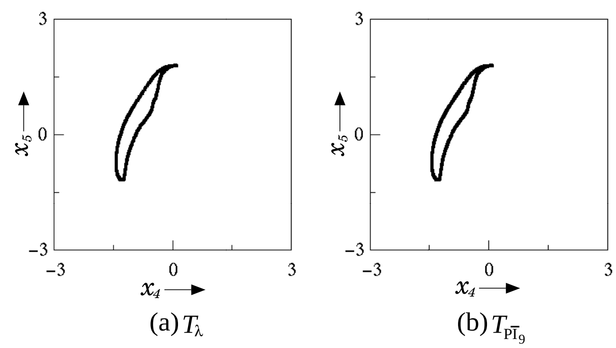

Fig. 6. Points of (a) Poincaré mapping T_λ and (b) the mapping $T_{P\bar{I}_9}$ for the symmetrical three-phase quasi-periodic solution.

region we observed a stable symmetrical three-phase periodic solution. By decreasing the value of ω_0 from this region, the Neimark–Sacker bifurcation N_1 occurs and the quasi-periodic solution (Fig. 5) is generated.

Figure 6 shows points of the Poincaré mapping T_λ and the mapping $T_{P\bar{I}_9}$. From Fig. 6, we can see that this quasi-periodic solution satisfies the definition of symmetrical three-phase given in Sec. 2.3; therefore, Fig. 5 shows a symmetrical three-phase quasi-periodic solution.

3.3. *Asymmetrical three-phase quasi-periodic and chaotic solutions*

Figure 7 shows a bifurcation diagram of the three-phase periodic solutions observed in Eq. (1) with $\omega = 0.5$. We observed a stable three-phase solution

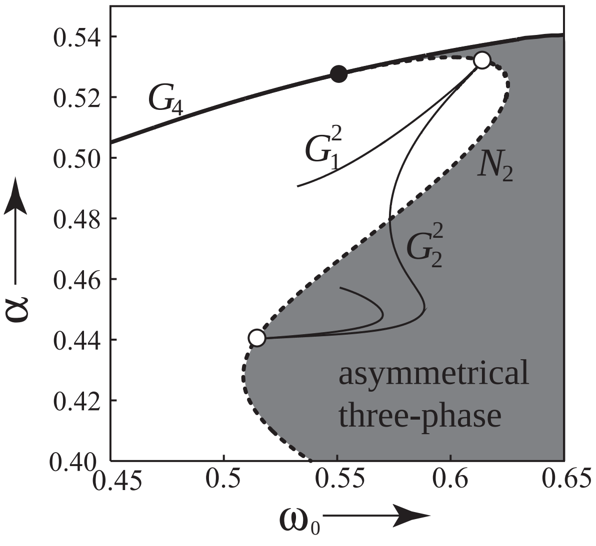

Fig. 7. Bifurcation diagram of asymmetrical three-phase periodic solutions observed in Eq. (1) with $\omega = 0.5$.

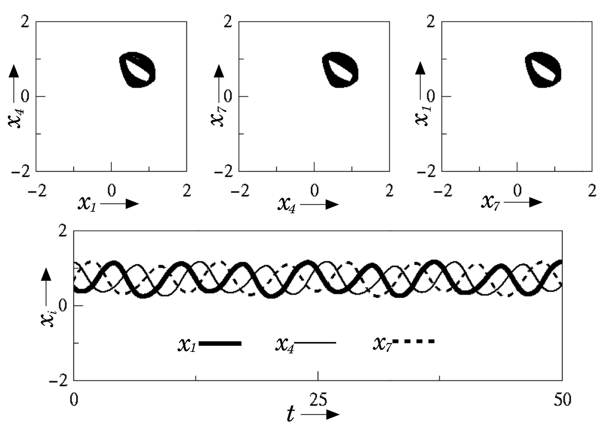

Fig. 8. Three-phase quasi-periodic solution: $\alpha = 0.497$, $\omega_0 = 0.6$, $\omega = 0.5$.

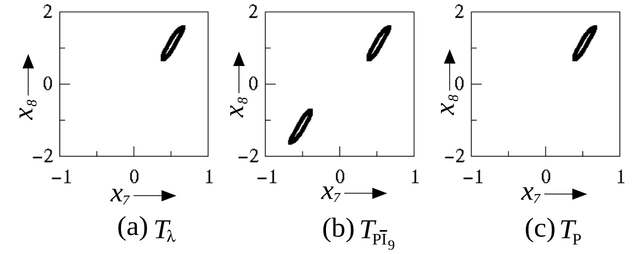

Fig. 9. Points of (a) Poincaré mapping T_λ, (b) the mapping $T_{P\bar{I}_9}$, and (c) the mapping T_P for the three-phase quasi-periodic solution.

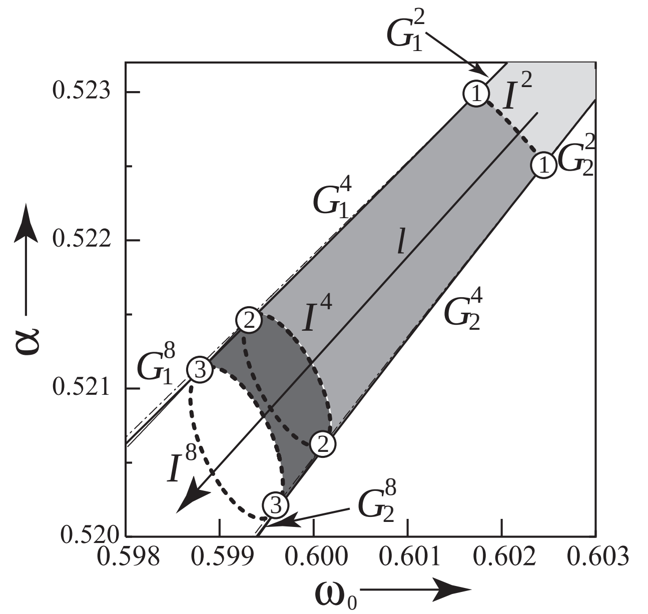

Fig. 10. Enlarged bifurcation diagram of a part of Fig. 7. The points marked by ①, ②and ③indicate codimension-two bifurcation point, called TP bifurcation.

in the region shaded by ▬. By the Neimark–Sacker bifurcation N_2, the stable three-phase solution becomes unstable and the quasi-periodic solution (Fig. 8) is generated. This quasi-periodic solution does not satisfy the definition of symmetrical three-phase [Fig. 9(b)]; however, it satisfies the definition of asymmetrical three-phase [Fig. 9(c)]. Thus, we call it a three-phase quasi-periodic solution.

In Fig. 7, the tangent bifurcation sets G_1^2 and G_2^2 meet the Neimark–Sacker bifurcation set as cusp points at the points marked by open circles (the argument of the characteristic multipliers equals π radian). We only show the tangent bifurcation set of two-periodic solutions. However, the tangent bifurcation sets of various kinds of periodic solutions were also observed [Mihara & Kawakami, 1996; Kitajima & Kawakami, 1997], which are called the Arnold tongues [Arnold, 1983].

Figure 10 shows an enlarged bifurcation diagram of a part of Fig. 7. In the region surrounded by the tangent bifurcation sets G_1^2 and G_2^2, the period-doubling bifurcation set I^2 of a two-periodic solution, generated by the tangent bifurcations, exist. The intersecting points of the tangent bifurcation sets and the period-doubling set are codimension-two bifurcation points, called the TP bifurcation [Yoshinaga & Kawakami, 1995]. From the points marked by ①, the tangent bifurcation sets G_1^4 and G_2^4 of four-periodic solutions appear. At the points marked by ②, codimension-two bifurcation (TP bifurcation) occur again via the intersection of G_1^4, G_2^4 and I^4. Moreover, by the same mechanism,

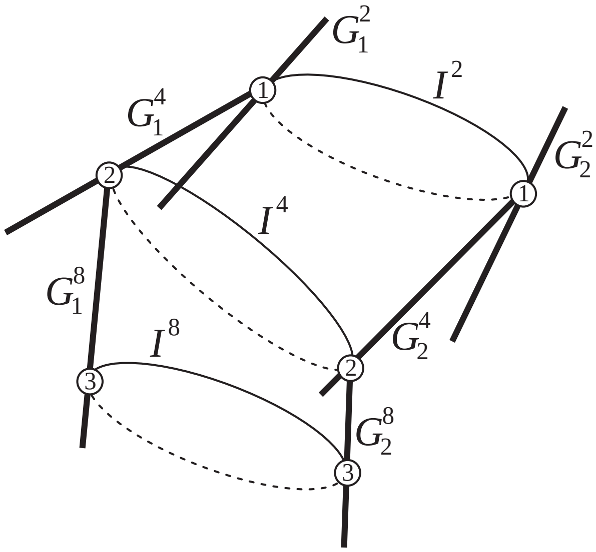

Fig. 11. Schematic bifurcation diagram of successive TP bifurcations.

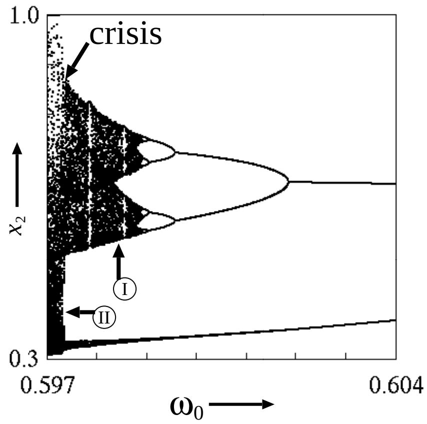

Fig. 12. One-parameter bifurcation diagram.

successive TP bifurcations occur and chaotic solution is generated. The schematic diagram is shown in Fig. 11. We observed tree-like pattern of the tangent bifurcation sets, discussed in [Yoshinaga & Kawakami, 1989].

Figure 12 shows a one-parameter bifurcation diagram changing the parameter values along the line l in Fig. 10. From this figure, we can confirm that the chaotic state is generated by successive period-doubling bifurcations. The chaotic solution, at the parameter value marked by Ⓘ, does not satisfy the definition of asymmetrical three-phase (see Fig. 13). However, after the symmetry-increasing crisis [Melbourne *et al.*, 1993], it satisfies the definition of asymmetrical three-phase (see Fig. 14). Thus, we can observe the asymmetrical three-phase

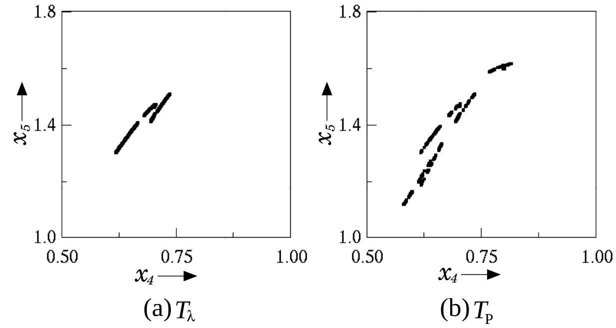

Fig. 13. Points of (a) Poincaré mapping T_λ and (b) the mapping T_P for the non-three-phase chaotic solution: $\omega = 0.5$, $\alpha = 0.52078$, $\omega_0 = 0.5985$.

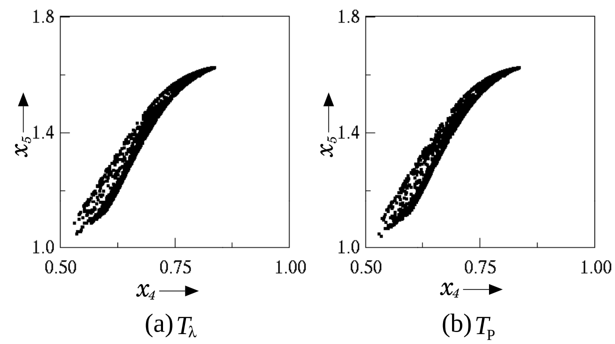

Fig. 14. Points of (a) Poincaré mapping T_λ and (b) the mapping T_P for the three-phase chaotic solution: $\omega = 0.5$, $\alpha = 0.51993$, $\omega_0 = 0.5972$.

chaotic solution at the parameter value marked by Ⓘ in Fig. 12.

4. Concluding Remarks

We have investigated the phase transition between the solutions with distinct symmetrical property observed in a system of coupled three-oscillators with hard characteristics and state coupling. By a symmetry-breaking bifurcation, a symmetrical in-phase solution bifurcates into synchronized modes, called a partially in-phase solution and an almost in-phase solution. We have summarized the results of possible phase transitions observed in the symmetrical system with respect to the dihedral group D_3 and the inversion of state variables.

Moreover, by using the definition of symmetrical and asymmetrical three-phase solutions, we confirmed the existence of a stable symmetrical three-phase quasi-periodic solution and an asymmetrical three-phase chaotic solution. We have clarified the bifurcation mechanism of generating the three-phase chaotic solution: successive period-doubling bifurcations of phase locked state (asymmetrical solution) occur and the three-phase chaotic solution appears after symmetry-increasing bifurcation.

In coupled oscillator systems with symmetrical properties, it is an interesting open problem to find the universal symmetrical property of periodic solutions generated by phase locking phenomena.

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