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LETTER

Forced Synchronization of Coupled Oscillators

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SUMMARY We consider a system of coupled two oscillators with external force. At first we introduce the symmetrical property of the system. When the external force is not applied, the two oscillators are synchronized at the opposite phase. We obtain a bifurcation diagram of periodic solutions in the coupled system when the single oscillator has a stable anti-phase solution. We find that the synchronized oscillations eventually become in-phase when the amplitude of the external force is increased.

key words: *coupled oscillator, bifurcation, symmetry*

1. Introduction

Systems of coupled oscillators have been used extensively in physiological and biochemical modeling studies. Using group theoretic discussion applied to the coupled oscillators, we can derive some general theorems concerning with the bifurcations of equilibrium points and periodic solutions [1]. Many investigators have been studied two mutually coupled oscillators [2]–[4](mutual synchronization) because two oscillators' case is a prototype modeling to understand the phenomena in a large number of coupled oscillators. For instance, Kimura et al. investigated synchronization phenomena observed in two oscillators coupled by a resistor with current connection [4]. They confirmed that these oscillators were synchronized in the opposite phase. On the other hand forced synchronization is also studied in the filed of physiology (forced BVP [5]), chemistry (forced Brusselator [6]) and electric engineering (forced van der Pol [7]). However we cannot find the study of connecting mutual synchronization and forced synchronization.

In this study a forced coupled oscillator is analyzed. The dynamics of the circuit becomes invariant under the transformations: (1) interchange of the state variables, and (2) inversion of state variables with time shift π radian [8]. The periodic external force is injected into the invariant subspace of the transformation (1). When the external force is not applied, the two oscillators are synchronized in the opposite phase. We obtain a bifurcation diagram of periodic solutions in the coupled system when the single oscillator has a stable anti-phase solution. We find that the synchronized oscillations eventually become in-phase when the amplitude of the external force is increased. The bifurcation processes corresponding to the synchronizations stated above are clarified by the bifurcation diagram. In the diagram we obtain codimension three bifurcation points of intersection of D-type of branching and Neimark-Sacker bifurcation. Around these points we observe bifurcations of quasi-periodic solutions.

These results are useful for understanding the complicated phenomena in a simple forced oscillator [9] and are important for the study of biological rhythm and biorhythm like ultradian rhythm and circadian rhythm.

2. Circuit Equation and Related Property

We assume nonlinear conductance $g(v)$ and voltage source $e(t)$ in Fig. 1 as

$$g(v) = -a_1 v + a_3 v^3, \quad e(t) = E\sin(\nu t). \tag{1}$$

Then the normalized circuit equations are described by

$$\begin{aligned}
\frac{dr_1}{dt} &= -\left[-c_1 + \frac{c_3}{2}({r_1}^2 + 3{r_2}^2)\right] r_1 - \omega s_1 - \delta_1 r_1 \\
&\quad + \sqrt{2}\delta_1 B\sin(\nu t) \\
\frac{ds_1}{dt} &= \omega r_1 - \sigma s_1 \\
\frac{dr_2}{dt} &= -\left[-c_1 + \frac{c_3}{2}(3{r_1}^2 + {r_2}^2)\right] r_2 - \omega s_2 - \delta_2 r_2 \\
\frac{ds_2}{dt} &= \omega r_2 - \sigma s_2
\end{aligned} \tag{2}$$

where

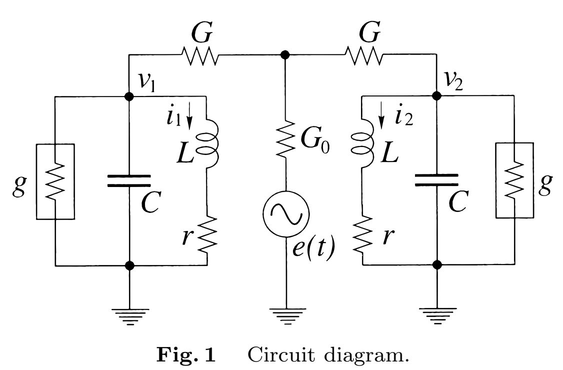

Fig. 1 Circuit diagram.

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$$c_1 = \frac{a_1}{C},\ c_3 = \frac{a_3}{C^2},\ \sigma = \frac{r}{L},\ \omega = \frac{1}{\sqrt{LC}},$$
$$B = \sqrt{C}E,\ \delta_1 = \frac{G}{C(1+2GG_0)},\ \delta_2 = \frac{G}{C},$$
$$x_i = \sqrt{C}v_i,\ y_i = \sqrt{L}i_i, \quad (i = 1, 2)$$

and

$$\begin{bmatrix} r_1 & s_1 \\ r_2 & s_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \tag{3}$$

Equations (2) have following symmetrical operations:

$$\begin{aligned}
&\sigma_0 : R^4 \times R \to R^4 \times R\ ;\\
&\quad (r_1\ s_1\ r_2\ s_2\ \nu t) \mapsto (r_1\ s_1\ -r_2\ -s_2\ \nu t)\\
&I_{1/2} : R^4 \times R \to R^4 \times R\ ;\\
&\quad (r_1\ s_1\ r_2\ s_2\ \nu t) \mapsto (-r_1\ -s_1\ -r_2\ -s_2\ \nu t - \pi)\\
&\sigma_{1/2} : R^4 \times R \to R^4 \times R\ ;\\
&\quad (r_1\ s_1\ r_2\ s_2\ \nu t) \mapsto (-r_1\ -s_1\ r_2\ s_2\ \nu t - \pi)
\end{aligned} \tag{4}$$

3. Method of Analysis

We assume the periodic solution in Eq. (2) as

$$x(t) = \varphi(t, x, \lambda) \tag{5}$$

where

$$\varphi(0, x, \lambda) = x \tag{6}$$

and define the Poincaré map:

$$T:\ R^4 \to R^4;\ x \longmapsto T(x) = \varphi(2\pi/\nu, x, \lambda). \tag{7}$$

Then the fixed point x_0 of T satisfies

$$x_0 - T(x_0) = 0 \tag{8}$$

and the characteristic equation is described as

$$\chi(\mu) = \det(\mu I_4 - DT(x_0)) \tag{9}$$

where

$$DT(x_0) = \frac{\partial T(x_0)}{\partial x} = \frac{\partial \varphi(2\pi/\nu, x, \lambda)}{\partial x}, \tag{10}$$

$\mu = 1$: tangent bifurcation,

$\mu = 1$: D-type of branching (degenerate case),

$\mu = e^{j\theta}$: Neimark-Sacker bifurcation.

We can obtain bifurcation parameters of the fixed point solving Eqs. (8) and (9) simultaneously [10].

4. Results

We fix the parameters in Eq. (2) as

$$c_3 = 1/3, \quad \delta_1 = 1.0, \quad \omega = 1.0, \quad \sigma = 0.5.$$

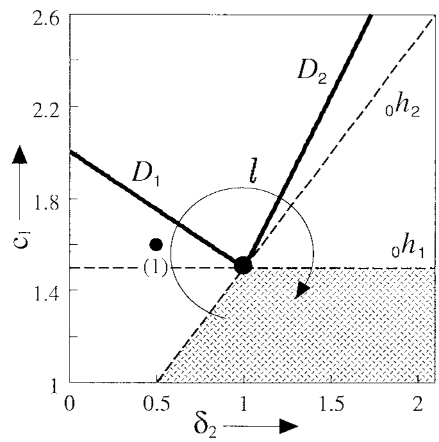


Fig. 2 Bifurcation diagram of equilibrium points and periodic solutions in Eq. (2) where $B = 0$. Dashed and solid lines indicate Hopf bifurcation of the equilibrium point at the origin and D-type of branching of periodic solutions, respectively.

4.1 Bifurcations of the Unforced System

At first we study bifurcations of the unforced system with $B = 0$ in Eq. (2). Figure 2 shows a bifurcation diagram of the unforced system. In the shaded region there exists a stable equilibrium point at the origin. Changing the parameters along the curve l, the first Hopf bifurcation ${}_0h_2$ and the second Hopf bifurcation ${}_0h_1$ generate the anti-phase and the in-phase periodic solution, respectively. The in-phase solution meets the D-type of branching D_1 (symmetry-breaking bifurcation) and generates two $I_{1/2}$-invariant solutions. In the next section, we consider that the unforced system has stable anti-phase and unstable in-phase solutions (the point marked by (1) in Fig. 2). Note that $\delta_2 < 1$ for $\delta_1 = 1$ implies $GG_0 < 0$ in the original circuit.

4.2 Forced Synchronization

Figure 3 shows a bifurcation diagram of periodic solutions in Eq. (2). Because the unforced system ($B = 0$) has anti-phase and in-phase solutions, their corresponding bifurcation sets G_1, G_2, and G_4, respectively, meet the axis of $B = 0$ at ν_1 and ν_2. Here we are interested in how to change the anti-phase solution under the influence of external force. Figures 4 (a)–4 (d) show trajectories of the solutions when the amplitude B of the external force is increased. Note that the external force is applied to the in-phase direction (see Eq. (2)). When $B = 0$ the oscillators synchronize in the opposite phase, see Fig. 4 (a). Increasing B, two $I_{1/2}$-invariant solutions appear (Figs. 4 (b) and (c)). If we can find one of them by the operation σ_0 or $\sigma_{1/2}$, we can easily obtain the other solution. In the region these two solutions stably exist. Increasing B the oscillators synchronize with in-phase in the region .

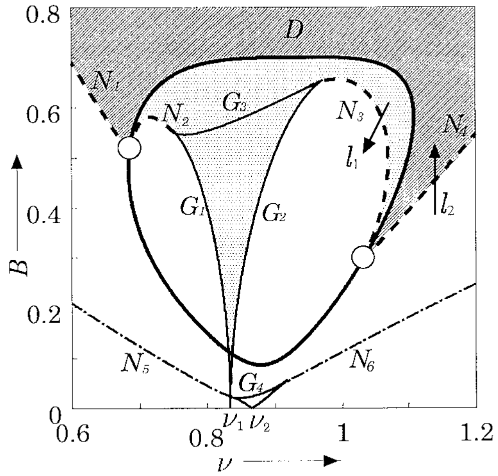
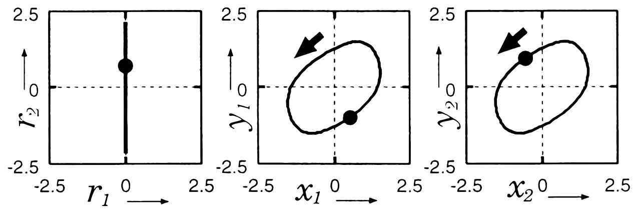
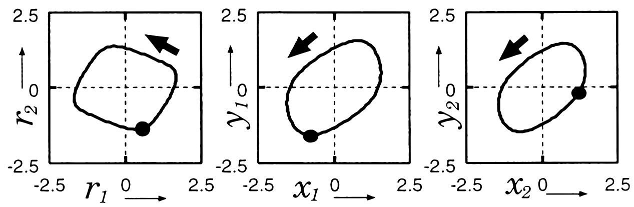


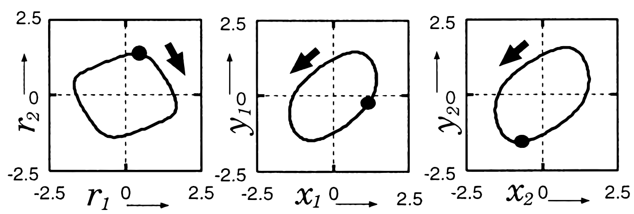
Fig. 3 Bifurcation diagram of periodic solutions in Eq. (2). The symbols G and N indicate tangent and Neimark-Sacker bifurcation set, respectively. $\delta_2 = 0.5$. $c_1 = 1.6$.



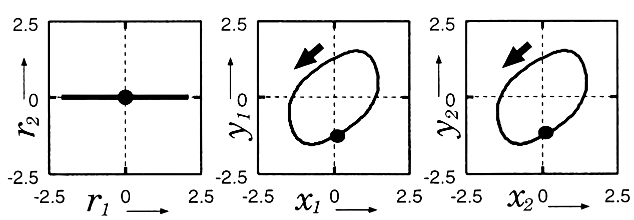
(a) $B = 0$. Anti-phase.



(b) $\nu = 0.8$. $B = 0.6$. $I_{1/2}$-invariant.

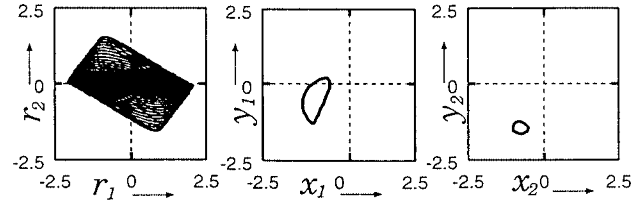


(c) $\nu = 0.8$. $B = 0.6$. $I_{1/2}$-invariant.

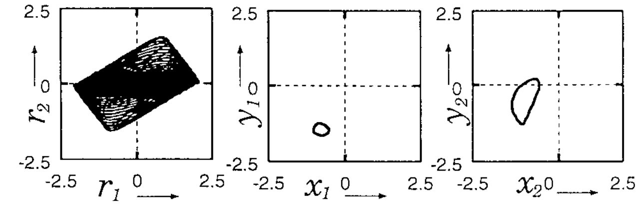


(d) $\nu = 0.8$. $B = 0.8$. In-phase.

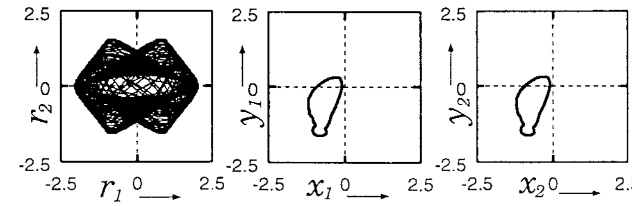
Fig. 4 Trajectories of the solutions in Eq. (2). Arrows and the points marked by closed circles indicate the time direction of the trajectory and the fixed point of Poincaré map, respectively. (Left) r_1 vs. r_2. (Middle) Oscillator 1. (Right) Oscillator 2.



(a) $\nu = 1.0$. $B = 0.627$.



(b) $\nu = 1.0$. $B = 0.627$.



(c) $\nu = 1.0$. $B = 0.615$.

Fig. 5 Quasi-periodic solutions in Eq. (2). (Left) Trajectories. r_1 vs. r_2. (Middle) The points of Poincaré map for Oscillator 1. (Right) The points of Poincaré map for Oscillator 2.

In Fig. 3 open circles indicate the points of intersection of Neimark-Sacker bifurcation set and D-type of branching set of periodic solutions called codimension three bifurcation. Two Neimark-Sacker bifurcation sets N_1 and N_2 (or N_3 and N_4) are the bifurcations of different periodic solutions. Changing the parameter along the line l_1 and l_2, two stable and one unstable quasi-periodic solutions are generated, respectively. The two stable solutions are shown in Figs. 5 (a) and (b). Decreasing the parameter B, these two quasi-periodic solutions become one solutions (see Fig. 5 (c)).

This bifurcation structure of quasi-periodic solutions is similar to that of P^2-codimension two bifurcation point [11]. Thus we can predict that the large quasi-periodic solution disappears the unstable quasi-periodic solution generated by Neimark-Sacker bifurcation N_4.

5. Concluding Remarks

We have investigated synchronization of coupled two oscillators with external force. When the external force is not applied, the two oscillators are synchronized in the opposite phase. We obtain the bifurcation diagram of periodic solutions in the coupled system when the single oscillator has a stable anti-phase solution. We found that the synchronized oscillations eventually become in-phase when the amplitude of the external force is increased.

The future problems are to study as follows:

- forced synchronization when the unforced system has different periodic solution,
- the bifurcation of quasi-periodic solutions around the points of intersection of D-type of branching and Neimark-Sacker bifurcation.

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