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Codimension Two Bifurcation Observed in a Phase Converter Circuit

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SUMMARY   We investigate bifurcations of the periodic solution observed in a phase converter circuit. The system equations can be considered as a nonlinear coupled system with Duffing's equation and an equation describing a parametric excitation circuit. In this system there are two types of solutions. One is with $x = y = 0$ which is the same as the solution of Duffing's equation (correspond to uncoupled case), another solution is with $xy \neq 0$. We obtain bifurcation sets of both solutions and discuss how does the coupling change the bifurcation structure. From numerical analysis we obtain a codimension two bifurcation which is intersection of double period-doubling bifurcations. Periodic solutions generated by these bifurcations become chaotic states through a cascade of codimension three bifurcations which are intersections of D-type of branchings and period-doubling bifurcations.
key words: *codimension two bifurcation, codimension three bifurcation, coupled system, nonlinear dynamical system*

1. Introduction

In this paper we study codimension two bifurcations observed in a forced oscillatory circuit containing saturable inductors, which is called a phase converter circuit, see Fig. 1 and Ref. [1]. The normalized circuit equations are described by

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -ky - (c_0 + c_1)x - \frac{1}{4}c_3(x^2 + 3u^2)x$$
$$\frac{du}{dt} = v \qquad\qquad (1)$$
$$\frac{dv}{dt} = -kv - \left(\frac{1}{3}c_0 + c_1\right)u - \frac{1}{4}c_3(x^2 + 3u^2)u$$
$$\qquad\qquad + B\cos t + B_0$$

where

$$x = \phi_a - \phi_b, \quad u = \phi_c, \quad k = g/\omega C, \quad B \propto E, \quad B_0 \propto E_0.$$

In this circuit, the characteristic of inductor is assumed to be a cubic function, i.e., the relation between current i and flux ϕ is assumed to be $i = c_1\phi + c_3\phi^3$. Equations (1) can be considered as a nonlinear coupled system with Duffing's equation and an equation describing

a parametric excitation circuit. Hence we may observe various complicated behavior such as chaotic state due to a period-doubling process of parametric excitation, codimension two bifurcations, the coexistence of several periodic oscillations which are correlated with jump and hysteresis behaviors and the fundamental, higher-harmonic and sub-harmonic resonances.

In particular we are interested in codimension two bifurcations. In the neighborhood of a codimension two bifurcation point the dynamical behavior exhibits complicated features and some types of codimension two bifurcations may relate to the generation of chaotic states. Numerical analysis shows that a new route to chaos through a cascade of codimension three bifurcations.

2. Method of Analysis

Equations (1) are rewritten as:

$$\dot{x} = f(t, x, \lambda) \qquad\qquad (2)$$

In Eqs. (1), f is periodic in t with period 2π:

$$f(t + 2\pi, x, \lambda) = f(t, x, \lambda) \qquad\qquad (3)$$

We assume that Eqs. (1) has a solution $x(t) = \varphi(t, x_0, \lambda)$ with initial condition $x_0 : x(0) = \varphi(0, x_0, \lambda) = x_0$. Since f has the period 2π, we can naturally define the Poincaré mapping T_λ from the state space R^4 into itself:

$$T_\lambda : R^4 \to R^4; x_0 \longmapsto T_\lambda(x_0) = \varphi(2\pi, x_0, \lambda) \qquad (4)$$

If a solution $\varphi(t, x_0, \lambda)$ is a periodic, then the point x_0 is a fixed point of T_λ:
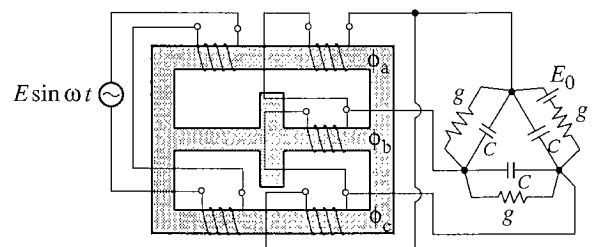
$$F_\lambda(x_0) := x_0 - T_\lambda(x_0) = 0 \qquad\qquad (5)$$

Fig. 1   A phase converter circuit.

Computation of a periodic solution of Eqs. (1) has now been reduced to find $x_0 \in R^4$ satisfying Eq. (5).

The generic bifurcations of the periodic solution are known as codimension one bifurcations: Tangent, period-doubling, Neimark-Sacker bifurcations. At a bifurcation value of parameters, if a periodic solution satisfies two bifurcation conditions, then the bifurcation refers to as a codimension two bifurcation. The degenerate bifurcation: D-type of branching observed in the system which possess a symmetrical property is also called codimension two bifurcation. These bifurcation sets can be traced out by solving the fixed point equation and bifurcation condition simultaneously [2].

We use the notation $_k D^m$ (resp. $_k I^m$) denoting a type of m periodic points of T_λ having even (resp. odd) number of characteristic multipliers on the real axis $(-\infty, -1)$, and k indicates the number of characteristic multiplier outside the unit circle in the complex plane.

3. Results

Now we show some numerical results of bifurcation diagrams and behavior of the solutions observed in Eqs. (1). We use the following notations in bifurcation diagrams: G_k^m, I_k^m and D_k^m represent respectively tangent, period-doubling bifurcation and D-type of branching of m-periodic point and k denotes the number to distinguish several bifurcation sets of the same period.

3.1 Decision of the System Parameters

Firstly we explain how to decide the system parameters in Eqs. (1). In the following we fix the several parameters in Eqs. (1) as

(a) $c_0 = c_1 = 0.0$, $c_3 = 1.0$,

(b) $k = 0.1$.

The reasons are as follows:

(a) We choose the same value in [2],

(b) We obtain 3-dimensional bifurcation diagram shown in Fig. 2. When k is lager than 0.3, the intersection of period-doubling bifurcations disappear. If k is less than 0.1, bifurcation structure would be more complicated.

Thus we consider the bifurcation problem in the parameter plane for B and B_0 with $k = 0.1$.

3.2 Bifurcation of Uncoupled Case

Secondly we show a bifurcation diagram, see Fig. 3, of the solution with $x = y = 0$, i.e., the solution of Duffing's equation of u and v. Thus the diagram is the same as that of Duffing's equation. In the shaded region, there
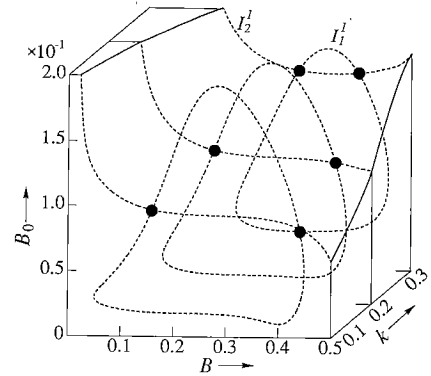
Fig. 2 Bifurcation diagram in (B, B_0, k) space. Dashed curves and circled points indicate period-doubling bifurcations and codimension two bifurcations, respectively.
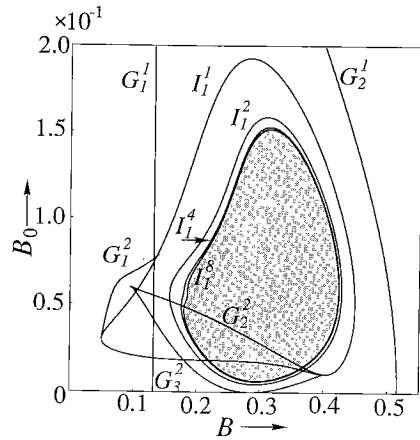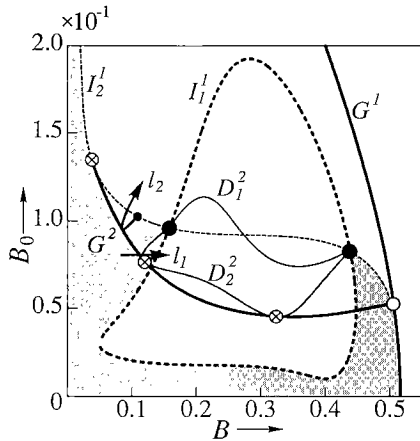

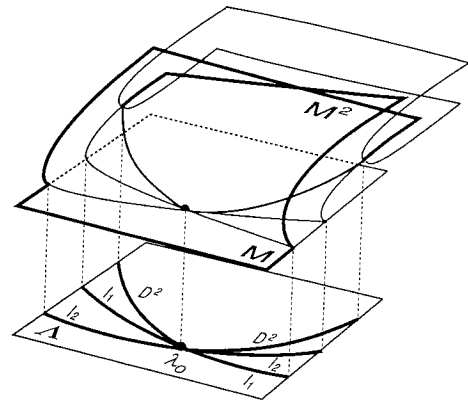**Fig. 3** Bifurcation diagram of the periodic points of T_λ with $x = y = 0$.

exists chaotic state by a cascade of period-doubling bifurcations. In the next section we will describe how does the coupling change this bifurcation structure.
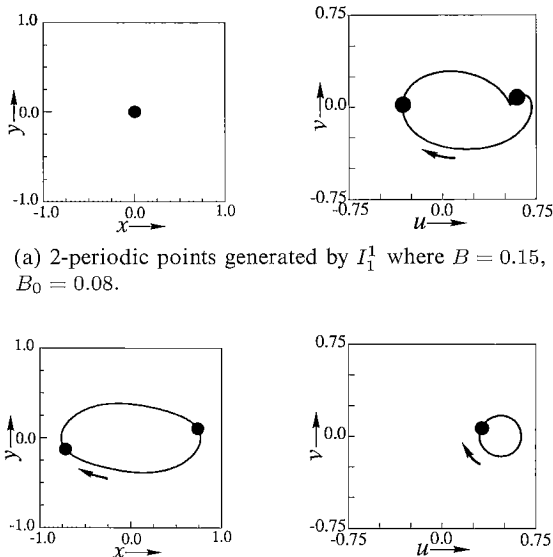
3.3 Bifurcation of Coupled Case

Thirdly we show the change of the bifurcation structure under the influence of coupling term. In Fig. 4 a stable fixed point with $x = y = 0$ exists in the shaded region ▨ . By increasing the parameter B along the line l_1, 2-periodic points with $x = y = 0$ are generated on the period-doubling bifurcation curve I_1^1, see Fig. 5 (a). This bifurcation with $x = y = 0$ is therefore the same as the bifurcation of Duffing's equation shown in Fig. 3. On the other hand when B and B_0 change along the curve l_2, the period-doubling bifurcation I_2^1 generates 2-periodic points with $xy \neq 0$, see Fig. 5 (b), which corresponds to the parametric excitation phenomenon in the original circuit. At black circles in Fig. 4 the two period-doubling bifurcations I_1^1 and I_2^1 intersect, which are called codimension two bifurcations of double period-doubling bifurcations [3]. In the neighborhood of this point D-type of branchings

Fig. 4 Bifurcation diagram for Eqs. (1). Open circle indicate the codimension two bifurcation called TP-bifurcation in [7]. From this point tangent bifurcation of 2-periodic points appear.

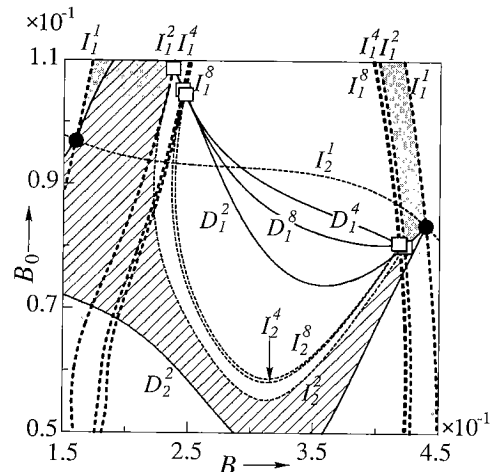(a) 2-periodic points generated by I_1^1 where $B = 0.15$, $B_0 = 0.08$.

(b) 2-periodic points generated by I_2^1 where $B = 0.12$, $B_0 = 0.1$.

Fig. 5 Trajectories generated by the period-doubling bifurcations: I_1^1 and I_2^1. The points marked ● and the arrow indicate the 2-periodic points of T_λ and the direction of trajectory, respectively.

D_2^2 and D_1^2 necessarily appear, see Fig. 6 [4].

Figure 7 shows the detail bifurcation diagram of Fig. 4. In the figure we see successive codimension three bifurcations which are the intersections of D-type of branchings and period-doubling bifurcations. The mechanism of generation of successive codimension three bifurcations are as follows:

(1) A cascade of period-doubling bifurcation $I_1^{2^n}$ ($n = 1, 2, 3$) of 2^n periodic points with $x = y = 0$ exists, see Fig. 3. When $x = y = 0$, Eqs. (1) reduces to Duffing's equation of u and v as mentioned Sect. 3.2.

Fig. 6 Manifold of fixed point M and manifold of 2-periodic points M^2 in the neighborhood of λ_0: Intersection of double period-doubling bifurcations. In the manifold heavy and light curves represent stable and unstable manifolds.


Fig. 7 Detailed bifurcation diagram of Fig. 4. The symbols square indicate intersection of D-type of branching and period-doubling bifurcation which is called codimension three bifurcation. Two cascades of codimension three bifurcations are observed.
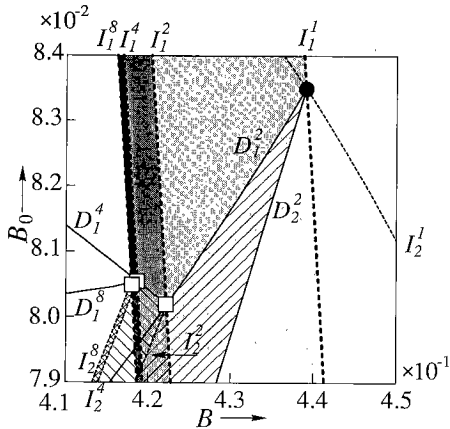
(2) Under the influence of coupling term another period-doubling bifurcation I_2^1 appears. At the intersecting point of I_1^1 and I_2^1, codimension two bifurcation is generated, and from this intersecting point D-type of branchings D_1^2 and D_2^2 appear.

(4) Bifurcation curves $D_1^{2^n}$ and $I_1^{2^n}$ intersect at square marked points in Fig. 7 which are codimension three bifurcations. In the neighborhood of this point, D-type of branching of 2^{n+1}- periodic points $D_1^{2^{n+1}}$ and period-doubling bifurcation of 2^n-periodic points $I_2^{2^n}$ necessarily appear, see Fig. 8. Similarly $D_1^{2^{n+1}}$ and $I_1^{2^{n+1}}$ will intersect on the cascade process of period-doubling bifurcation.

(5) By repeating (4) the successive codimension three bifurcations occur and finally become chaotic states.
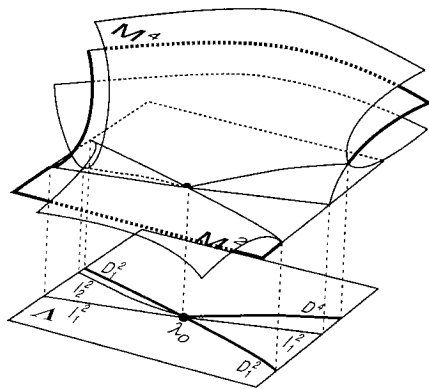
Fig. 8 Manifolds of 2-periodic points M^2 and manifolds of 4-periodic points M^4 in the neighborhood of λ_0: Intersection of period-doubling bifurcation and D-type of branching. Λ indicates parameter plane. In the manifold heavy and light curves represent stable and unstable manifolds.

Fig. 10 Schematic diagram of phase transitions observed in Fig. 9. Bifurcations written by italic letters and shades are correspond to those of Fig. 9. Solid and dashed lines indicate stable and unstable periodic points, respectively.
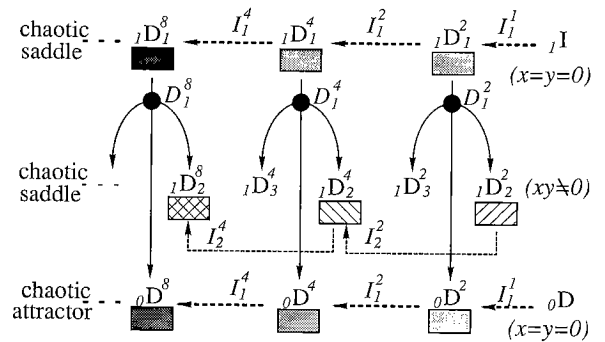

Fig. 9 Enlarged bifurcation diagram of Fig. 7.

In Fig. 9 we see that bifurcation set D^{2^n} forms a zigzag branch. A similar tree-like pattern is found in [5] on a period-doubling cascade of one-dimensional dynamical system. This property can be explained by a renormalization of a family of functions on the cascade [6]. In Fig. 9 we observe three types of phase transitions to chaos, see Fig. 10. One of them is from a stable periodic point to a chaotic attractor. Others are from periodic points of saddle type to chaos. The latter chaotic states may have both stable and unstable invariant manifolds. Hence the chaotic state can be called a chaotic saddle and it is unstable.

4. Concluding Remarks

We have investigated global properties of bifurcation sets of periodic solutions observed in the phase converter circuit. This system is a coupled system of Duffing's circuit and a parametric excitation circuit. We show how does the coupling change the bifurcation structure. The main results of our analysis are as fol-

lows: We obtain a codimension two bifurcation which is intersection of double period-doubling bifurcations. Periodic solutions generated by these bifurcations become chaotic states through a cascade of codimension three bifurcations which are intersections of D-type of branchings and period-doubling bifurcations.

Further research is needed to study a detailed mechanism of generating the double period-doubling bifurcations and classification of the codimension three bifurcations.

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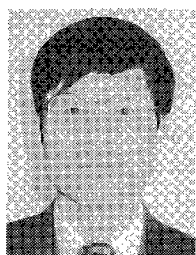
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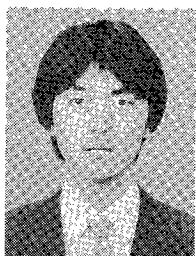
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