PAPER

Synchronized States Observed in Coupled Four Oscillators

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SUMMARY In this paper, we examine oscillatory modes generated by the Hopf bifurcations of equilibrium points except for the origin in a system of coupled four oscillators. (The bifurcation analyses of the origin for many coupled oscillators were already done.) The Hopf bifurcations of the equilibrium points with strong symmetrical property and the generated oscillatory modes are classified. We observe four-phase, in-phase and a pair of anti-phase synchronized states. Even in a system of four coupled oscillators, we discover the existence of a stable three-phase oscillation. By the numerical bifurcation analysis of generated periodic oscillations we find out successive period-doubling bifurcations as the route to chaos and show some of them are symmetry-breaking bifurcations. As a result of the symmetry-breaking period-doubling bifurcations, a periodic solution with complete synchronization becomes a chaotic solution with phase synchronization.
key words: *coupled oscillator, bifurcation, phase synchronization, symmetry*

1. Introduction

Systems of coupled oscillators are widely used as models of biological rhythmic oscillations such as human circadian rhythms [1], [2], finger movements [3], animal locomotion [4], swarms of fireflies that flash in synchrony, synchronous firing of cardiac pacemaker cells [5], [6], and so on. Using these coupled oscillator models, many investigators have studied the mechanism of generation of synchronous oscillations and phase transitions between distinct oscillatory modes. From the standpoint of the bifurcation, the former and the latter correspond to the Hopf bifurcation of an equilibrium point (or the tangent bifurcation of a periodic solution) and the pitch-fork bifurcation (or the period-doubling bifurcation) of a periodic solution, respectively. Using group theoretic discussion applied to the coupled oscillators, we can derive some general theorems concerning with the bifurcations of equilibrium points and periodic solutions [7].

In the study of the coupled oscillator system, the four-coupled oscillator system is one of the most interesting system, because there exists an irregular degenerate oscillatory mode (or an independent pair of anti-phase mode) [8], [9] when the equation of the single oscillator is invariant under inversion of state variables. Mishima and Kawakami studied the oscillatory modes generated by the Hopf bifurcations of the origin (equilibrium point) in several systems of coupled four BVP (Bonhöffer-van der Pol) oscillators [10]. However, they considered the Hopf bifurcation of the origin, because it is only supercritical. Tsumoto et al. investigated bifurcations of the Modified BVP (MBVP) equation [11]. In the MVBP system, the supercritical Hopf bifurcation of non-origin equilibrium points occurs.

In this paper, we examine the oscillatory modes generated by the Hopf bifurcations of non-origin equilibrium points in the four-coupled oscillator system. The Hopf bifurcations of the equilibrium points with strong symmetrical property and the generated oscillatory modes are classified. We observe four-phase, in-phase and a pair of anti-phase synchronized states. Even in a system of four coupled oscillators, a stable three-phase oscillation is also observed. By the numerical bifurcation analysis of generated periodic oscillations we find out successive period-doubling bifurcations as the route to chaos and show some of them are symmetry-breaking bifurcations. As a result of the symmetry-breaking period-doubling bifurcations, a periodic solution with complete synchronization becomes a chaotic solution with phase synchronization.

This paper is organized as follows. In Sect. 2, circuit equations are shown. In Sect. 3, we present the results of our study; classification of synchronized states and their bifurcations. Finally, in Sect. 4 we summarize our results as conclusion.

2. Circuit Equations

We consider the system of coupled four MBVP oscillators shown in Fig. 1. The four oscillators are globally coupled by resistors, because it is the simplest way to obtain many stable synchronized states. The circuit equations are described

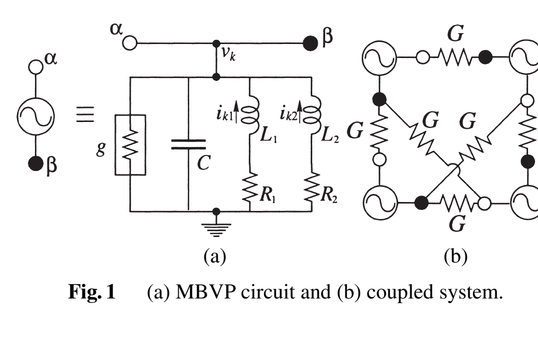

Fig. 1 (a) MBVP circuit and (b) coupled system.

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as

$$\begin{aligned} L_1\frac{di_{k1}}{dt} &= -R_1 i_{k1} - v_k \\ L_2\frac{di_{k2}}{dt} &= -R_1 i_{k2} - v_k \\ C\frac{dv_k}{dt} &= i_{k1} + i_{k2} - g(v_k) \\ &\quad - G(3v_k - v_{k+1} - v_{k-1} - v_{k+2}) \\ (k = 1,&\cdots,4, v_0 \equiv v_4, v_5 \equiv v_1, v_6 \equiv v_2), \end{aligned} \tag{1}$$

where the nonlinear conductance $g(v_k)$ is assumed to be

$$g(v_k) = -v_k + \frac{1}{3}v_k^3. \tag{2}$$

The values of system parameters are fixed as [11]

$$L_1^{-1} = 0.2,\ R_1 = 4.0,\ R_2 = 2.1,\ C^{-1} = 3.0 \tag{3}$$

for the occurrence of the supercritical Hopf bifurcation of non-origin equilibrium points.

The Jacobi matrix of Eq. (1) is described by

$$DF = \begin{bmatrix} X_0 & X_1 & X_1 & X_1 \\ X_1 & X_0 & X_1 & X_1 \\ X_1 & X_1 & X_0 & X_1 \\ X_1 & X_1 & X_1 & X_0 \end{bmatrix}. \tag{4}$$

Each block is given by

$$\begin{aligned} X_0 &= \begin{bmatrix} -R_1L_1^{-1} & 0 & -L_1^{-1} \\ 0 & -R_2L_2^{-1} & -L_2^{-1} \\ C^{-1} & C^{-1} & C^{-1}(1-v_{*k}^2) - 3d \end{bmatrix}, \\ X_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d \end{bmatrix} \end{aligned} \tag{5}$$

where v_{*k} is an equilibrium point and $d = C^{-1}G$. Using orthogonal matrix given by

$$Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1/\sqrt{2}I & I & O & 1/\sqrt{2}I \\ 1/\sqrt{2}I & O & I & -1/\sqrt{2}I \\ 1/\sqrt{2}I & -I & O & 1/\sqrt{2}I \\ 1/\sqrt{2}I & O & -I & -1/\sqrt{2}I \end{bmatrix} \tag{6}$$

where I is 3×3 identity matrix and O is 3×3 zero matrix, we diagonalize the Jacobian matrix (4) as

$$Q^{-1}\cdot DF\cdot Q = \begin{bmatrix} Y_0 & O & O & O \\ O & Y_1 & O & O \\ O & O & Y_1 & O \\ O & O & O & Y_1 \end{bmatrix} \tag{7}$$

where

$$Y_0 = X_0 + 3X_1, \tag{8}$$
$$Y_1 = X_0 - X_1. \tag{9}$$

In the next section we classify the oscillatory modes generated by the Hopf bifurcation in each block Y_l $(l = 0,\ 1)$ in Eq. (7).

3. Results

In the single MBVP oscillator two equilibrium points a and $-a$ have the supercritical Hopf bifurcation. Considering the coupled system described by Eq. (1), the equilibrium points with the supercritical Hopf bifurcation are the combination of the two equilibrium points a and $-a$; those are $(v_{*1}, v_{*2}, v_{*3}, v_{*4}) = (a, a, a, a), (a, a, -a, -a)$ and $(a, a, a, -a)$. When $G \neq 0$ the last type cannot exist, thus we use the notation (a_3, a_3, a_3, b) $(b \neq -a_3)$.

In the next subsection we show Hopf bifurcations of above three types of equilibrium points and classify generated oscillatory patterns.

3.1 (a_1, a_1, a_1, a_1) type

In Fig. 2 we show Hopf bifurcation sets in the parameter plane (d, L_2^{-1}). The stable equilibrium point is observed in the shaded regions. By changing the value of the parameter L_2^{-1} from the stable regions and crossing the Hopf bifurcations of Y_0, then an in-phase oscillation occurs. On the other hand a four-phase oscillation is generated by passing through the Hopf bifurcation of Y_1, see Fig. 3. Note that in waveforms shown in hereafter, there is no correspondence

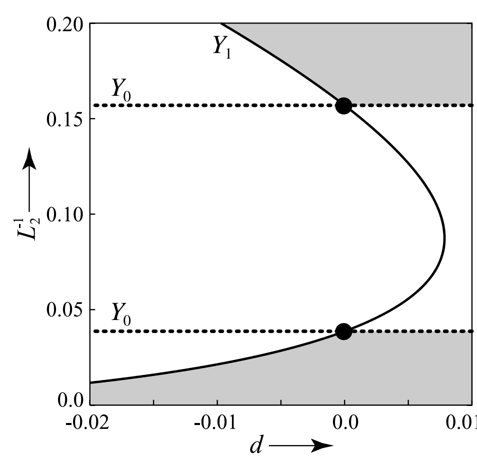

Fig. 2 Hopf bifurcations of block Y_0 and Y_1 for (a_1, a_1, a_1, a_1) type equilibrium point. Closed circles indicate codimension-two bifurcation points called Hopf-Hopf bifurcation [12].

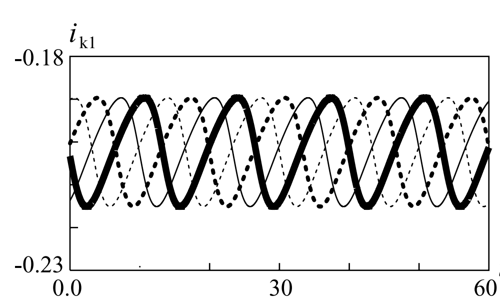

Fig. 3 Waveforms of a four-phase synchronized state in Eq. (1) with $L_2^{-1} = 0.035$ and $d = -0.01$.

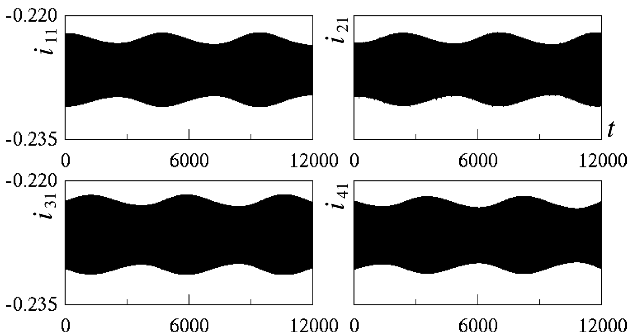

Fig. 4 Waveforms of a four-phase quasi-periodic solution in Eq. (1) with $L_2^{-1} = 0.035$ and $d = -0.017$.

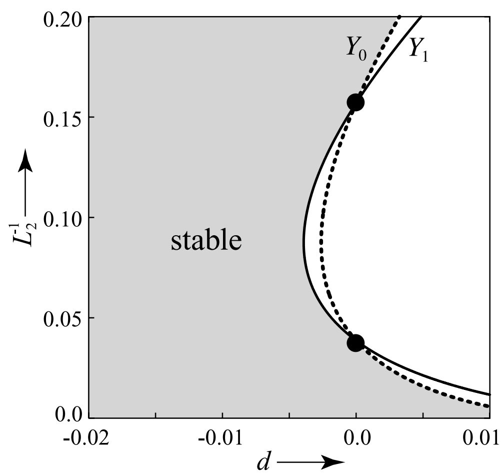

Fig. 5 Hopf bifurcations of block Y_0 and Y_1 for $(a_2, a_2, -a_2, -a_2)$ type.

between the kinds of curved line and the order of oscillators denoted by k, because the oscillators are fully connected and we can exchange any oscillators. This four-phase solution meets the Neimark-Sacker bifurcation and a four-phase quasi- periodic solution appears as shown in Fig. 4.

3.2 $(a_2, a_2, -a_2, -a_2)$ type

We show a bifurcation diagram for $(a_2, a_2, -a_2, -a_2)$ type equilibrium point in Fig. 5. Hopf bifurcations of Y_0 and Y_1 generate a pair of in-phase and a pair of anti-phase oscillations shown in Figs. 6 and 7, respectively. The one-parameter bifurcation diagram for the in-phase solution (Fig. 8) represents the occurrence of successive period-doubling bifurcations by changing the value of the parameter L_2^{-1}. At $L_2^{-1} = 0.059$, 0.060 and 0.061 we observe the two-periodic (Fig. 9), the four-periodic (Fig. 10) and the chaotic oscillation (Fig. 11), respectively.

The first period-doubling bifurcation is called symmetry-breaking bifurcation because the waveforms of the generated two-periodic solution are slightly different, see Fig. 9(a). However the waveforms of Fig. 9(b) are the same, because this symmetry ($i_{31} = i_{41}$) is not broken. By the next period-doubling bifurcation the symmetry ($i_{31} = i_{41}$) is broken, see Fig. 10(b). Note that the waveforms of

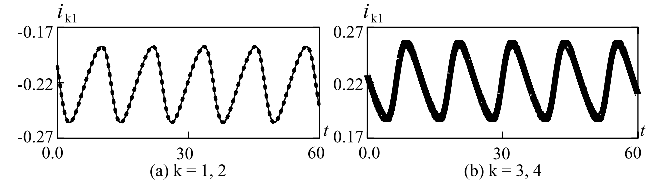

Fig. 6 Waveforms of a pair of in-phase oscillations in Eq. (1) with $L_2^{-1} = 0.058$ and $d = -0.0001$.

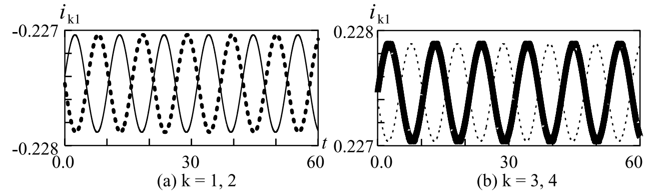

Fig. 7 Waveforms of a pair of anti-phase synchronized states in Eq. (1) with $L_2^{-1} = 0.048$ and $d = -0.0016$.

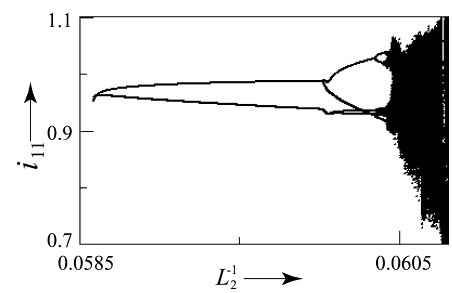

Fig. 8 One-parameter bifurcation diagram of a pair of anti-phase oscillations in Eq. (1) with $d = -0.0001$.

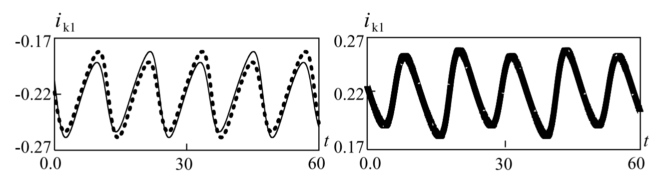

Fig. 9 Waveforms of two-periodic solution generated by period-doubling bifurcation of Fig. 6 in Eq. (1) with $L_2^{-1} = 0.059$ and $d = -0.0001$.

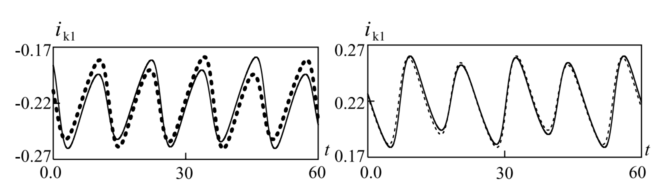

Fig. 10 Waveforms of four-periodic solution generated by period-doubling bifurcation of Fig. 9 in Eq. (1) with $L_2^{-1} = 0.060$ and $d = -0.0001$.

Fig. 10(a) are different, because this symmetry ($i_{11} = i_{21}$) is already broken by the first period-doubling bifurcation. After successive period-doubling bifurcations, chaotic oscillation shown in Fig. 11 appears. From this figure we can see that the complete synchronization is broken, however the

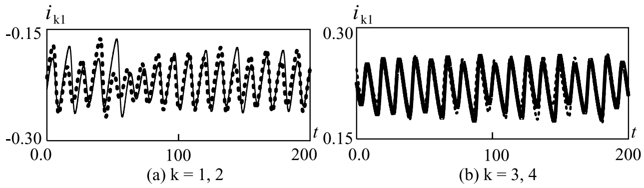

Fig. 11 Waveforms of chaotic oscillation in Eq. (1) with $L_2^{-1} = 0.061$ and $d = -0.0001$.

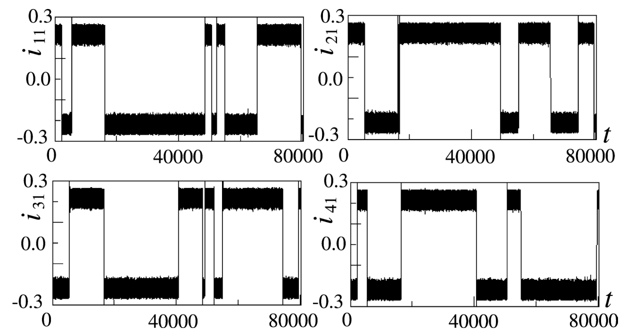

Fig. 12 Switching phenomenon of chaotic oscillation in Eq. (1) with $L_2^{-1} = 0.062$ and $d = -0.0001$.

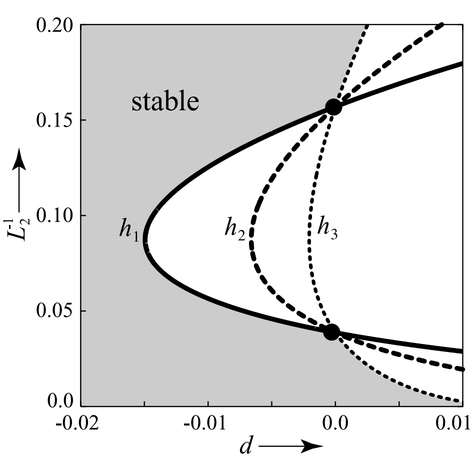

Fig. 13 Hopf bifurcation sets of (a_3, a_3, a_3, b) type in Eq. (1).

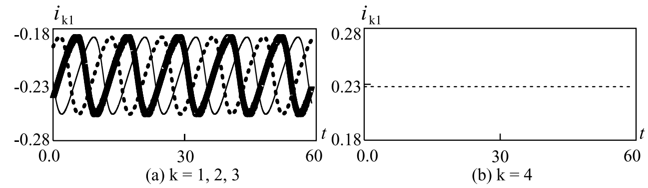

Fig. 14 Waveforms of a 3-phase synchronized state in Eq. (1) with $L_2^{-1} = 0.06$ and $d = -0.0016$.

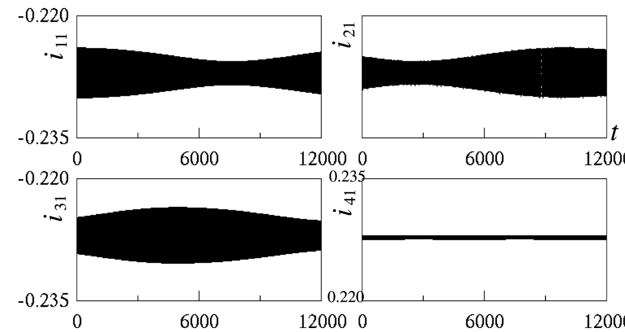

Fig. 15 Waveforms in Eq. (1) with $L_2^{-1} = 0.04$ and $d = -0.0015$.

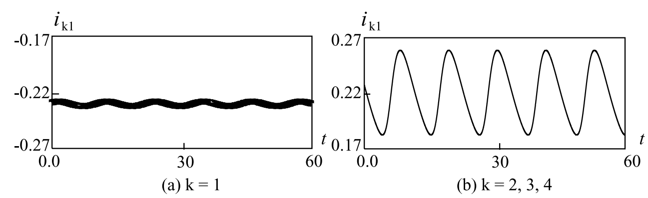

Fig. 16 Waveforms of an in-phase synchronized state caused by the Hopf bifurcation of (a_3, a_3, a_3, b) in Eq. (1) with $L_2^{-1} = 0.0711$ and $d = -0.0033$.

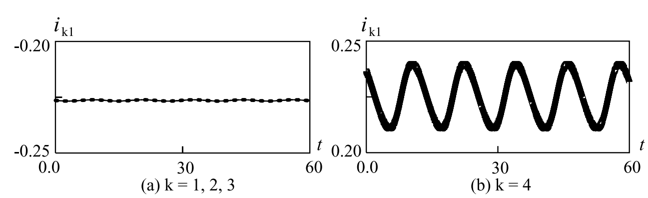

Fig. 17 Waveforms of an in-phase synchronized state caused by the Hopf bifurcation of (a_3, a_3, a_3, b) in Eq. (1) with $L_2^{-1} = 0.038$ and $d = 0.0005$.

phase synchronization [13] is kept.

Moreover, by increasing the value of the parameter L_2^{-1}, the switching phenomenon of this chaotic oscillation is observed as shown in Fig. 12. The switch occurs for two or four oscillators simultaneously, thus always the currents i_{k1} of two oscillators are positive and those of the others are negative. We suppose that this phenomenon may correlate to global structure of stable and unstable manifolds of a saddle type periodic solution embedded in the chaotic attractor. Detailed analysis is one of our future problems.

3.3 (a_3, a_3, a_3, b) type

Figure 13 shows Hopf bifurcation sets of (a_3, a_3, a_3, b) type equilibrium point. By the Hopf bifurcation denoted by h_1, oscillation of Fig. 14 is generated. From this figure we can see that three oscillators synchronized at three-phase (Fig. 14(a)) and the other oscillation is almost stopped (Fig. 14(b)). The reason why we can observe the almost stop phenomenon is that the sum of three oscillatory waves is almost constant. By changing the values of parameters L_2^{-1} and d, the three-phase oscillation meets the Neimark-Sacker bifurcations, and the three-phase quasi-periodic solution appears as shown in Fig. 15. This result agrees with the result of Kuznetsov [12]; near the Hopf-Hopf bifurcation there exists a resonant solution.

On the other hand, by the Hopf bifurcation denoted by h_2 and h_3, in-phase oscillations of three oscillators shown in Figs. 16 and 17 appear, respectively. In these figures the amplitude of in-phase oscillations of three oscillators is differ-

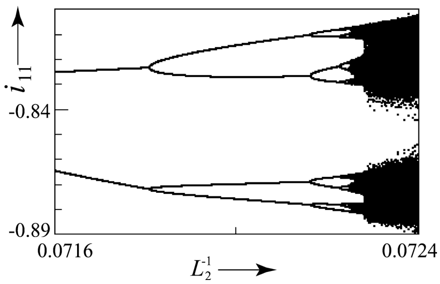

Fig. 18 One-parameter bifurcation diagram of the in-phase solution in Eq. (1) with $d = -0.0033$.

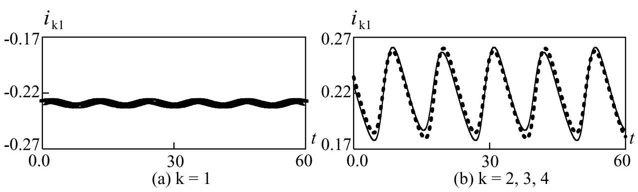

Fig. 19 Waveforms of an in-phase synchronized state in Eq. (1) with $L_2^{-1} = 0.0716$ and $d = -0.0033$.

Table 1 Classification of oscillatory modes.

equilibrium point	oscillatory modes
(a_1, a_1, a_1, a_1)	4-phase, in-phase(1)
$(a_2, a_2, -a_2, -a_2)$	a pair of anti-phase, a pair of in-phase
(a_3, a_3, a_3, b)	3-phase, a pair of in-phase

ent; large (Fig. 16(b)) and small (Fig. 17(a)). The in-phase oscillation shown in Fig. 16 is unstable after passing through h_2, but by the Neimark-Sacker bifurcation it becomes stable. By changing the value of the parameter L_2^{-1} successive period-doubling bifurcations of this in-phase solution shown in Fig. 18 are observed. By the first period-doubling bifurcation, the waveform of one oscillator becomes different (Fig. 19(b)) and in-phase synchronization of three oscillators is broken. However, this symmetry (in-phase of two oscillators) is kept by the other successive period-doubling bifurcations, thus chaotic oscillation with this symmetry is generated.

3.4 Summary

In Table 1 we classify oscillatory modes generated by the Hopf bifurcations of three types of equilibrium points. For each equilibrium point, the in-phase solution is generated by the Hopf bifurcation in the block Y_0 in Eq. (7). On the other hand, the Hopf bifurcations in the blocks Y_1 are degenerate (three pairs of eigenvalues satisfy the condition of the Hopf bifurcation). Thus many oscillatory modes (stable and unstable) appear simultaneously by the degenerate Hopf bifurcation.

We illustrate an interesting oscillatory mode generated by the degenerate Hopf bifurcation in Fig. 20. In this figure the oscillations of two oscillators are completely stopped and the others are synchronized at anti-phase. The sum of two oscillatory waves is zero, thus the others can be equilibrium states.

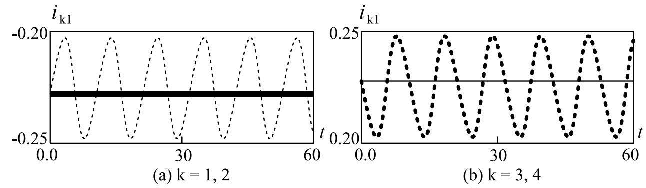

Fig. 20 Waveforms of oscillation death (unstable) in Eq. (1) with $L_2^{-1} = 0.06$ and $d = -0.0023$.

4. Conclusion

We have studied the oscillatory modes generated by the Hopf bifurcations in coupled four oscillators. The Hopf bifurcations of three types of equilibrium points and the generated oscillatory modes are classified. Moreover, by numerical bifurcation analysis we observed various interesting synchronized states caused by the degenerate Hopf bifurcations. Considering the associative memory model for storing patterns as oscillatory states [14], this system has the advantage of many oscillatory modes.

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