Saddle-node bifurcation parameter detection strategy with nested-layer particle swarm optimization

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Abstract

Nested-layer particle swarm optimization (NLPSO) detects bifurcation parameters in discrete-time dynamical systems. Previous studies have proven the effectiveness of NLPSO for period-doubling bifurcations, but not for other bifurcation phenomena. This paper demonstrates that NLPSO can effectively detect saddle-node bifurcations. Problems in detecting saddle-node bifurcation parameters by conventional NLPSO are clarified, and are solved by imposing a simple condition on the NLPSO objective function. Under this conditional objective function, the NLPSO accurately detected both saddle-node and period-doubling bifurcation parameters regardless of their stability, without requiring careful initialization, exact calculations or Lyapunov exponents.

Keywords: Bifurcation point detection, bifurcation analysis, initial value setup problem, discrete-time dynamical systems, particle swarm optimization (PSO)

1. Introduction

Complex real-world dynamics or phenomena can be modeled as dynamical systems. Dynamical systems research covers a wide range, embracing engineer-
ing, biology, sociology, ecology, and other fields. Most dynamical systems contain one or more parameters and exhibit nonlinear behavior. As the parameters are varied, the periodic solutions of the system may undergo a sudden qualitative change called a bifurcation. Moreover, small changes in the parameter values may largely affect the system behavior. Bifurcation analysis, which investigates how bifurcations depend on the system parameters, is therefore among the most important nonlinear analysis techniques for understanding system phenomena.

Bifurcation parameters can be found by various methods [1, 2, 3], and many software packages, such as AUTO [4] and BunKi [5], are available for computing bifurcation diagrams. However, these conventional methods are based on the Newton–Raphson method, which is a gradient-based algorithm requiring derivation of the system equations and appropriate initial values. Such methods can be difficult for beginners who are unfamiliar with nonlinear analysis.

Alternatively, population-based optimization is a metaheuristic optimization consisting of multiple potential solutions. A popular algorithm is particle swarm optimization (PSO) [6], which avoids the derivation of the objective functions. However, although many PSO-based periodic-point detection [7, 8, 9, 10, 11] and bifurcation analysis [12, 13] methods have been proposed, they cannot detect two or more bifurcation parameters directly and easily, and therefore lack versatility.

To overcome this limitation, Matsushita et al. proposed a PSO-based method that directly detects period-doubling bifurcation parameters [14]. This method, called nested layer particle swarm optimization (NLPSO), is performed by two nested PSOs. The main PSO searches for the period-doubling bifurcation parameters, whereas the inner PSO searches for the periodic point using the system parameter, which corresponds to the information of one agent in the main PSO. Using the periodic point returned by the inner PSO, the main PSO calculates the objective function for that agent under the bifurcation conditions. Although the NLPSO requires no carefully set initial system parameters or exact calculations, it quickly and accurately finds the period-doubling bifurcation parameter without a Lyapunov or gradient-based method, regardless of the stability of the periodic point. The initial value problem of conventional bifurcation analyses
can then be solved by applying the NLPSO-detected bifurcation parameter to the initial point of the bifurcation curve tracing. The NLPSO detects the parameters of period-doubling bifurcations with high accuracy, strong robustness, versatile usability and rapid convergence speed. Therefore, its application to bifurcation parameters other than period-doubling bifurcation must be tested.

This paper demonstrates that NLPSO detects saddle-node bifurcation parameters under a simple condition of its objective function. We first investigate the detection of saddle-node bifurcation parameters by NLPSO. Under the original objective-function conditions, the NLPSO fails to detect the saddle-node bifurcation parameter correctly because its inner loop fails the periodic-point search, yet the algorithm finishes the bifurcation parameter search. To resolve this problem, we then impose a new condition on the NLPSO objective function. Under this new condition, the objective function extends the NLPSO to the detection of saddle-node bifurcation parameters without affecting its ability to detect period-doubling bifurcation parameter. The NLPSO with the conditional objective function was applied to two discrete-time dynamical systems: a circle map and a Hénon map [15]. In these experiments, the NLPSO accurately detected not only the period-doubling bifurcation parameters, but also the saddle-node bifurcation parameters, directly from the parameter space. Finally, we applied the NLPSO to a neuronal system [16, 17], a complex dynamical system with saddle-node bifurcations at high-periodical points. The simulation results confirmed proper detection of the saddle-node bifurcation parameters by NLPSO, confirming that NLPSO is a powerful bifurcation-parameter detection method.

The remainder of this paper is organized as follows. Section 2 defines a discrete-time dynamical system and the bifurcation of its fixed and periodic points. Sections 3 and 4 overview the PSO and NLPSO algorithms, respectively. Section 5 clarifies the problem with saddle-node bifurcation parameter detection by NLPSO, and solves it by imposing a simple condition on the objective function. Section 6 illustrates the validity and applicability of the NLPSO through several examples. Section 7 concludes our work and identifies several
directions of future research in our agenda.

2. Definition

Let us consider the following $N$-dimensional discrete-time dynamical system:

$$ x(k + 1) = f(x(k), \lambda), $$

where $k$ denotes discrete time, and $x(k) \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}^L$ correspond to the state variables and system parameters, respectively.

Let $f^l$ denote the $l$-th iteration of $f$. A point $x_p \in \mathbb{R}^N$ is said to be an $n$-periodic point of $f$ if $x_p = f^n(x_p, \lambda)$ and $x_p \neq f^l(x_p, \lambda)$ for $l < n$. A 1-periodic point is referred to as a fixed point.

The Jacobian matrix of $f$ at the $n$-periodic point $x_p$ is described by

$$ Df^n(x_p, \lambda) = \prod_{l=0}^{n-1} \frac{\partial f}{\partial x(n-1-l)}(x(n-1-l), \lambda), $$

with characteristic equation

$$ \det (Df^n(x_p, \lambda) - \mu I_N) = 0, $$

where $x(0) = x_p$, $I_N$ denotes the $N \times N$ identity matrix, and $\mu$ is a characteristic multiplier of $Df^n(x_p, \lambda)$. The characteristic multipliers determine the local stability class of an $n$-periodic point. If all characteristic multipliers of $Df^n(x_p, \lambda)$ are inside the unit circle in the complex plane, then $x_p$ is a stable periodic point (SPP). If any characteristic multipliers of $Df^n(x_p, \lambda)$ are outside the unit circle, then $x_p$ is an unstable periodic point (UPP). Period-doubling and saddle-node bifurcations occur under the conditions $\mu = -1$ and $\mu = 1$, respectively.

3. Overview of PSO

The PSO is one of the simplest and most popular population-based evolutionary computation techniques. It gives multiple potential solutions, called
particles, each carrying position and velocity information. The position and velocity vectors of the $i$-th particle at iteration $t$ are represented by $z_i(t) = (z_{i1}, z_{i2}, \ldots, z_{iD}) \in \mathbb{R}^D$ and $v_i(t) = (v_{i1}, v_{i2}, \ldots, v_{iD}) \in \mathbb{R}^D$, respectively, where $i = 1, 2, \cdots, M$ ($M$ being the number of particles in the swarm), and $D$ corresponds to the dimension of the solution space. The position vector corresponds to an objective variable of the optimization. Initially, the PSO particles are randomly distributed in the search space, and each particle moves toward its personal best position $p_i = (p_{i1}, p_{i2}, \cdots, p_{iD})$ ($p_{best}$), which defines its best previous position, and the global best position $p_g = (p_{g1}, p_{g2}, \cdots, p_{gD})$ ($g_{best}$), defining the best $p_{best}$ with the best objective value among all particles. As $p_i$ and $p_g$ are updated at every iteration, $p_g(t)$ is the global optimum solution at iteration $t$. The elements of $v_i$ and $z_i$ in each dimension $d$ of particle $i$ are updated as follows [18]:

$$
\begin{align*}
v_{id}(t+1) &= w v_{id}(t) + r_1 c_1 (p_{id}(t) - z_{id}(t)) + r_2 c_2 (p_{gd}(t) - z_{id}(t)), \\
z_{id}(t+1) &= z_{id}(t) + v_{id}(t+1),
\end{align*}
$$

(4)

where $(d = 1, 2, \cdots, D)$, and $r_1$ and $r_2$ are two uniform random numbers taking different values between 0 and 1 in each dimension $d$. The inertia weight $w$ determines how much of the particle’s previous velocity is preserved. $c_1$ and $c_2$ are fixed positive acceleration coefficients; usually $c_1 = c_2$. In this study, these three parameters were set to their optimal default values [19], namely, $w = 0.729$ and $c_1 = c_2 = 1.494$. These processes repeat until the user-defined stop criterion is reached.

4. Nested Layer Particle Swarm Optimization (NLPSO)

NLPSO finds the system parameters $\lambda$ bifurcate that an $n$-periodic point in a dynamical system $f$. The NLPSO is performed by two PSO algorithms with nested structure. This section briefly explains the NLPSO algorithm and its detection of bifurcation parameters in discrete-time dynamical systems.

The main PSO (PSO$_{bif}$) seeks the bifurcation parameter $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_L)$. The PSO$_{bif}$ has a position vector $z_b = (z_{b1}, z_{b2}, \cdots, z_{bL})$ corresponding to an
objective variable $\lambda$, and its objective function $F_{\text{bif}}(z_b)$ is defined as

$$F_{\text{bif}}(z_b) = \left| \det \left( Df^n \left( x^*_p, z_b \right) - \mu I_N \right) \right| < C_{\text{bif}},$$

(5)

where $C_{\text{bif}}$ is the stop criterion corresponding to the optimization accuracy, and $x^*_p$ is deemed an $n$-periodic point detected by the inner PSO (PSO$_{pp}$), which will be described below. Equation (5) defines the bifurcation condition at the deemed $n$-periodic point $x^*_p$, satisfying Eq. (3).

To calculate Eq. (5), we must derive the $n$-periodic point $x_p$ depending on the system parameter $\lambda$ corresponding to the PSO$_{bif}$ particle position $z_b$. The inner PSO (PSO$_{pp}$) of NLPSO searches for the $M_{\text{bif}}$ periodic points given by the information of their respective $M_{\text{bif}}$ particles in the PSO$_{bif}$. A position vector of the PSO$_{pp}$ $z_p = (z_{p1}, z_{p2}, \cdots, z_{pN})$ corresponds to an objective variable $x_p$.

The objective function of the PSO$_{pp}$ is defined as

$$F_{\text{pp}}(z_p) = \| f^n(z_p, \lambda) - z_p \| < C_{\text{pp}},$$

(6)

where $C_{\text{pp}}$ is the stop criterion of PSO$_{pp}$, $\| \cdot \|$ denotes Euclidean distance, and $f^n(z_p, \lambda)$ corresponds to the state variable, which defines the $n$-th iteration of $f$ with initial point $z_p$ and parameter $\lambda$ (see Eq. (1)). Therefore, $F_{\text{pp}}(z_p, \lambda)$ is minimized at 0 when $z_p$ is exactly an $n$-periodic point (i.e., $z_p \equiv x_p$). Meanwhile, the $\text{gbest} z_{pg}$ detected by the PSO$_{pp}$ after searching is called a deemed $n$-periodic point $x^*_p$ because it is not guaranteed to be a true $n$-periodic point.

5. Problems of NLPSO and a condition for solving the problems

One goal of this study is detecting saddle-node bifurcation parameters by using NLPSO. First, this section clarifies a problem with detecting saddle-node bifurcation parameters by NLPSO. For this purpose, the NLPSO algorithm is applied to a one-dimensional discrete dynamical system called a circle map. We then add a condition to the objective function that corrects this problem.
5.1. Investigation of NLPSO behavior in saddle-node bifurcation parameter detection with the conventional objective function.

When $\mu$ is set to $-1$ in Eq. (5), the NLPSO algorithm directly and accurately detects the period-doubling bifurcation parameters, regardless of whether the periodic points are stable or unstable [14]. We therefore focus on a property of saddle-node bifurcations (namely, that one of the characteristic multipliers is unity), and try to detect saddle-node bifurcation parameters by setting $\mu = 1$.

The circle map is described by

$$x(k + 1) = f(x(k), \lambda) = \left( x(k) + \lambda_1 - \frac{\lambda_2}{2\pi} \sin 2\pi x(k) \right) \mod 1.0,$$

(7)

where $x(k)$ is a state variable, and $\lambda = (\lambda_1, \lambda_2)$ are system parameters. The objective function of the PSO$_{bif}$, which detects the bifurcation parameters of Eq. (7), is given by

$$F_{bif}(z_b) = \left| \frac{d f^n(x_p^*, \lambda)}{dx_p^*} - 1 \right| = \prod_{l=0}^{n-1} \left( 1 - z_{b2} \cos 2\pi x(n - 1 - l) \right) - 1,$$

(8)

where $z_b = (z_{b1}, z_{b2}) \equiv \lambda = (\lambda_1, \lambda_2)$; thus, $L = 2$. $x$ is a state variable calculated by Eq. (7) and its initial value $x(0)$ is the deemed $n$-periodic point $x_p^*$ depending on the parameter set $(\lambda_1, \lambda_2)$ corresponding to the particle information of PSO$_{bif}$. The objective function of the $x_p^*$ detected by PSO$_{pp}$, is given by

$$F_{pp}(z_p) = \left| f^n(z_{p1}, \lambda) - z_{p1} \right|,$$

(9)

where $z_p = z_{p1}$; thus, $N = 1$.

Table 1 summarizes the experimental parameters of PSO$_{bif}$ and PSO$_{pp}$, and Table 2 states the search-space ranges of the variables on the circle map. For different number of periodic points ($n = 2$, 3 and 5), we computed the saddle-node bifurcation parameters in 100 simulations with different random initial states of the particles. The respectively detected 100 bifurcation points (denoted $G^n$) are plotted in Fig. 1(a). Each of these points is supposed to denote
Table 1: Variables and parameters of the nested PSOs in NLPSO.

<table>
<thead>
<tr>
<th></th>
<th>PSO_{bif}</th>
<th>PSO_{pp}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of dimensions $D$</td>
<td>$L$</td>
<td>$N$</td>
</tr>
<tr>
<td>Stop criterion $C$</td>
<td>$1 \times 10^{-3}$</td>
<td>$1 \times 10^{-5}$</td>
</tr>
<tr>
<td>Maximum iterations $T_{\text{max}}$</td>
<td>300</td>
<td></td>
</tr>
<tr>
<td>Number of particles $M$</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>Inertia weight $w$</td>
<td>0.729</td>
<td></td>
</tr>
<tr>
<td>Acceleration coefficients $c_1, c_2$</td>
<td>1.494</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Search-space range of variables on the circle map.

<table>
<thead>
<tr>
<th>Variable</th>
<th>$\lambda_1 = z_{b_1}$</th>
<th>$\lambda_2 = z_{b_2}$</th>
<th>$x^*<em>p = z</em>{p_1}$</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[0.35, 0.65]</td>
<td>[0.5, 2]</td>
<td>[0, 1]</td>
<td></td>
</tr>
</tbody>
</table>

a saddle-node branching from an $n$-periodic point. Clearly, NLPSO failed to detect the saddle-node bifurcation parameters of any periodic point. Table 3 summarizes the parameters of 5 representative saddle-node bifurcations among the 100 detected bifurcations. Shown are the bifurcation parameters $(\lambda_1, \lambda_2)$, their accuracies $F_{\text{bif}}(\lambda)$, the iteration counts $t_{\text{end}}$ at which the $F_{\text{bif}}$ of $\text{gbest}$ satisfied the stop criterion $10^{-3}$, the $(\lambda_1, \lambda_2)$-dependent periodic points $x^*_p$ and their accuracies $F_{\text{pp}}(x^*_p)$. Although all trials in this table, reached the stop criterion $F_{\text{bif}} < 10^{-3}$, none of detected periodic points $x^*_p$ satisfied the stop criterion $F_{\text{pp}} < 10^{-5}$, and the PSO_{bif} stopped searching without once updating the particle position. To investigate the behavior of the PSO_{pp}, we extracted the parameters $(\lambda_1, \lambda_2) = (0.5447, 0.5551)$ from Table 3 and plotted their return map on the $x(k)$–$x(k + 2)$ plane. The results is shown in Fig. 1(b). As the PSO_{pp} particle position $z_{p_1}$ corresponds to the state variable $x(k)$ in the circle map, the particles move on the return map and search for a point $x(k)$ satisfying $x(k) = x(k + 2)$. Under the condition of the parameter set $\lambda$, the map confirms that no 2-periodic point satisfies $x(k) = x(k + 2)$. Therefore,
Figure 1: NLPSO results of saddle-node bifurcation parameter detection on the circle map using the conventional objective function given by Eq. (5). $\mu = 1$, $n = 2, 3$ or 5. (a) Detected 100 saddle-node bifurcation parameters for different numbers of periodic points. (b) Return map of the circle map with $(\lambda_1, \lambda_2) = (0.5447, 0.5551)$. The circle denotes the 2-periodic point $x_p = 0.5659$ obtained by the PSOpp.

The PSOpp returns the closest periodic point to $F_{pp} (x_p^*) = 0$. As this point lies outside the solution criterion ($10^{-5}$ of the true PSO_bif solution), it is called false periodic point. Thus, although the PSOpp did not find a correct a 2-periodic point, it returned a unity value of $Df^n (x_p^*, \lambda)$. Hence, for the parameters $\lambda$ and the false periodic point $x_p^*$, the value objective function $F_{bif}$ was $2.9512 \times 10^{-10}$, which satisfies the stop criterion $C_{bif}$. For that reason, the PSO_bif finished the parameter search without finding a correct saddle-node bifurcation parameter set. The NLPSO algorithm cannot solve this problem even when the maximum iteration $T_{\text{max}}$ is increased or the stop criteria $C_{bif}$ and $C_{pp}$ are set to stricter values. Furthermore, we note that this problem is particular to saddle-node bifurcations. In period-doubling bifurcation detection, if a PSO_bif particle takes system parameters $\lambda$ for which no $n$-periodic point exists, the PSOpp cannot detect a correct periodic point. Unlike saddle-node bifurcation detection, the
Table 3: Five representative parameters of saddle-node bifurcations for 2-, 3- and 5-periodic points in the circle map, detected by NLPSO using the conventional objective function.

<table>
<thead>
<tr>
<th>n</th>
<th>PSO_{bif}</th>
<th>PSO_{pp}</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda_1$</td>
<td>$\lambda_2$</td>
</tr>
<tr>
<td>2</td>
<td>0.5447</td>
<td>0.5551</td>
</tr>
<tr>
<td></td>
<td>0.4462</td>
<td>0.7765</td>
</tr>
<tr>
<td></td>
<td>0.6329</td>
<td>0.7492</td>
</tr>
<tr>
<td></td>
<td>0.5922</td>
<td>0.9747</td>
</tr>
<tr>
<td></td>
<td>0.6415</td>
<td>0.8116</td>
</tr>
<tr>
<td>3</td>
<td>0.361</td>
<td>0.5881</td>
</tr>
<tr>
<td></td>
<td>0.435</td>
<td>0.5758</td>
</tr>
<tr>
<td></td>
<td>0.4181</td>
<td>0.5763</td>
</tr>
<tr>
<td></td>
<td>0.4227</td>
<td>1.272</td>
</tr>
<tr>
<td></td>
<td>0.5662</td>
<td>0.7866</td>
</tr>
<tr>
<td>5</td>
<td>0.4596</td>
<td>0.911</td>
</tr>
<tr>
<td></td>
<td>0.3916</td>
<td>1.042</td>
</tr>
<tr>
<td></td>
<td>0.4645</td>
<td>1.456</td>
</tr>
<tr>
<td></td>
<td>0.4555</td>
<td>0.8382</td>
</tr>
<tr>
<td></td>
<td>0.636</td>
<td>0.8223</td>
</tr>
</tbody>
</table>

algorithm returns a bad $F_{bif}$ value that violates $F_{bif} < C_{bif}$. Therefore, the particle becomes attracted to particles with good $F_{bif}$ values (parameters for which an n-periodic point exists).

5.2. A condition that avoids “false periodic points”

As described above, NLPSO using the conventional $F_{bif}$ described in Eq. (5) fails to find a correct saddle-node bifurcation parameter. To avoid the detection
of false periodic points, we modify the objective function $F_{\text{bif}}$ of the PSO$_{\text{bif}}$ as

$$F_{\text{bif}}^*(z_b) = \begin{cases} \det \left( Df^n \left( x_p^*, z_b \right) - \mu I_N \right), & \text{if } F_{\text{pp}} \left( x_p^* \right) < C_{\text{pp}}, \\ \infty, & \text{otherwise.} \end{cases} \quad (10)$$

Equation (10) means that if the deemed periodic point $x_p^*$ detected by the PSO$_{\text{pp}}$ with $z_b_i$ is a false periodic point that violates $F_{\text{pp}} \left( x_p^* \right) < C_{\text{pp}}$, the $F_{\text{bif}}^*(z_b_i)$ of a particle $i$ is assigned a bad value ($F_{\text{bif}}^* = \infty$). In other words, when a PSO$_{\text{bif}}$ particle takes system parameters for which no $n$-periodic point exists, it does not affect the $p_{\text{best}}$ and $g_{\text{best}}$ of the PSO$_{\text{bif}}$. Therefore, the particles in the PSO$_{\text{bif}}$ converge to other particles taking system parameters for which $n$-periodic points do exist, and the $F_{\text{bif}}^*$ guarantees an accurate periodic point $x_p^*$ composed of the detected bifurcation parameters $\lambda$. Obviously, by setting $\mu = -1$, Eq. (10) can also detect period-doubling bifurcations with no negative effects. Although the added condition in $F_{\text{bif}}^*$ is very simple, it can solve the problem of detecting improper saddle-node bifurcation parameters. Equation (10) extends the NLPSO to saddle-node bifurcation parameter detection without complicating the algorithm.

6. Simulations

This section confirms that under the conditional objective function $F_{\text{bif}}^*$ described by Eq. (10), NLPSO detects not only period-doubling bifurcation parameters, but also saddle-node bifurcation parameters. To this end, the modified NLPSO is applied to various discrete-time dynamical systems.

6.1. Circle map

We first retry the detection of saddle-node bifurcation parameters in the circle map described by Eq. (7). We also confirm that the $F_{\text{bif}}^*$ effectively detects period-doubling bifurcation parameters. From Eqs. (7) and (10), the conditional
Figure 2: Parameter detection results of period-doubling bifurcations \( (I^n) \) and saddle-node bifurcations \( (G^n) \) in the circle map. All points were detected by the NLPSO using \( F_{\text{bif}}^* \), \( \mu = -1 \) or 1, \( n = 2, 3 \) or 5. (a) 100 bifurcation parameters were detected for each periodic number and each type of bifurcation. (b) Return map of the circle map with a saddle-node bifurcation parameter \((\lambda_1, \lambda_2) = (0.5248, 0.8093)\). The circle denotes the 2-periodic point \( x_p = 0.1639 \) obtained by the PSO\(_{pp}\).

The objective function of the PSO\(_{bif}\) is given by

\[
F_{\text{bif}}^*(z_b) = \begin{cases} 
\left| \frac{df}{dx}\left( x_p^*, z_b \right) \right| - \mu, & \text{if } F_{\text{pp}}(x_p^*) < C_{\text{pp}} \\
\infty, & \text{otherwise}
\end{cases}
\]

\[
= \begin{cases} 
\prod_{l=0}^{n-1} (1 - z_{b,2} \cos 2\pi x(n - 1 - l)) - \mu, & \text{if } F_{\text{pp}}(x_p^*) < C_{\text{pp}} \\
\infty, & \text{otherwise}
\end{cases}
\]

where \( x(0) = x_p^* \). Note that the PSO\(_{pp}\) objective function \( F_{\text{pp}} \) is given by Eq. (9).

The experimental PSO\(_{bif}\) and PSO\(_{pp}\) parameters, and the search ranges are those listed in Tables 1 and 2, respectively. The NLPSO detected a period-doubling bifurcation parameter \( (\mu = -1) \) and a saddle-node bifurcation param-
Figure 3: One-parameter bifurcation diagram of the circle map. The red line denotes the saddle-node bifurcation parameter detected by NLPSO with the conditional objective function $F_{bif}^*$. (a) $\lambda_1$ is varied, and the other parameter is fixed as $\lambda_2 = 0.8093$. (b) $\lambda_2$ is varied, and the other parameter is fixed as $\lambda_1 = 0.5248$.

Table 5 shows the results of 100 simulations initialized in different random states for each periodic point and type of bifurcation. The table shows the means and standard deviations (SDs) of $F_{bif}^*$ and $F_{pp}$ and the mean $t_{end}$ of the PSO$_{bif}$. In this table, the Suc[\%] values of the PSO$_{bif}$ and the PSO$_{pp}$ denote the success rates of $F_{bif}^* < C_{bif}$ and $F_{pp} < C_{pp}$, respectively. The PSO$_{bif}$ detected 100% of the period-doubling and saddle-node bifurcations, and the mean $t_{end}$ was well below the iteration limit ($T_{max} = 300$). Notably, the success
rate of the PSO<sub>pp</sub> was also 100%. Moreover, the small SDs of \( F_{\text{bif}}^* \) and \( F_{\text{pp}}^* \) confirm the high robustness of NLPSO with \( F_{\text{bif}}^* \). In Fig. 2(a), we plotted a two-dimensional bifurcation diagram of \( \lambda_1 \) versus \( \lambda_2 \), obtained by NLPSO with the conditional objective function \( F_{\text{bif}}^* \). The \( F_{\text{bif}}^* \) based NLPSO correctly obtained both the period-doubling bifurcation parameters and the saddle-node bifurcation parameters for all numbers of periods.

Let us consider the detected saddle-node bifurcation parameters when \( n = 2 \). Above we plotted the return map on the \( x(k) - x(k+2) \) plane for the parameter set \( z_b \equiv (\lambda_1, \lambda_2) = (0.5248, 0.8093) \) (Fig. 2(b)). The 2-periodic point \( x_p^* = 0.1639 \) satisfying \( F_{\text{pp}}^* (x_p^*) = 0 \) was correctly detected by the PSO<sub>pp</sub>. When a PSO<sub>bif</sub> particle takes parameters for which no 2-periodic point exists, and the PSO<sub>pp</sub> returns a false periodic point for that particle, and the PSO<sub>bif</sub> assigns a bad \( F_{\text{bif}}^* \) value (\( \infty \)) to the particle. PSO steers the poor-scoring particles toward those with good \( F_{\text{bif}}^* \) values, the PSO<sub>bif</sub> particles automatically converge to the parameter area containing the correct 2-periodic points of both bifurcation types, without requiring special manipulation or careful initialization.

### 6.2. Hénon map

We next consider the Hénon map, a two-dimensional map described by

\[
\begin{pmatrix}
    x_1(k+1) \\
    x_2(k+2)
\end{pmatrix} = f (\mathbf{x}, \lambda) =
\begin{pmatrix}
    1 - \lambda_1 x_1^2(k) + x_2(k) \\
    \lambda_2 x_1(k)
\end{pmatrix}
\]  

(12)

where \( \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \) and \( \lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2 \) are vectors containing the state variables and system parameters, respectively; thus, \( N = 2 \) and \( L = 2 \).

The conditional objective function of the PSO<sub>bif</sub>, which searches for bifur-
cation parameters, is defined as

\[
F_{\text{bif}}^*(z_b) = \begin{cases} 
\det \left( \frac{\partial f^n}{\partial x_p} (x_p^*, z_b) - \mu I_N \right), & \text{if } F_{\text{pp}} (x_p^*) < C_{\text{pp}} \\
\infty, & \text{otherwise,}
\end{cases}
\]

\[
= \begin{cases} 
\det \left( \prod_{l=0}^{n-1} \begin{bmatrix} -2z_{b1} x_1 (n-l) & 1 \\ z_{b2} & 0 \end{bmatrix} \right) - \mu \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \text{if } F_{\text{pp}} (x_p^*) < C_{\text{pp}}, \\
\infty, & \text{otherwise,}
\end{cases}
\]

(13)

where \( z_b = (z_{b1}, z_{b2}) \equiv \lambda = (\lambda_1, \lambda_2) \) and \( x_p^* = (x_{p1}^*, x_{p2}^*) \equiv (x_1(0), x_2(0)) \). The objective function of the PSO\(_{\text{pp}}\), which detects \((\lambda_1, \lambda_2)\)-dependent \(n\)-periodic points, is given by

\[
F_{\text{pp}} (z_p) = \| f^n (z_p, \lambda) - z_p \|,
\]

\[
= \sqrt{ (f_{z1}^n (z_p, \lambda) - z_{p1})^2 + (f_{z2}^n (z_p, \lambda) - z_{p2})^2 },
\]

(14)

where \( z_p = (z_{p1}, z_{p2}) \).

To compare the performances of the conditional objective function \( F_{\text{bif}}^* \) and the conventional objective function \( F_{\text{bif}} \), we applied both functions to the detection of period-doubling bifurcations \((\mu = -1)\) and saddle-node bifurcations \((\mu = 1)\) in the Hénon map with \(n\)-periodic points. The experimental NLPSO parameters were those listed in Table 1, and search ranges are summarized in Table 6. Period-doubling and saddle-node bifurcations within these ranges were confirmed in an earlier study [20].

Table 7 shows the results of 100 simulations of each bifurcation type and periodic number, and Fig. 4 is a two-dimensional bifurcation diagram of the 100 detected bifurcation parameter sets. The period-doubling bifurcation parameters obtained by NLPSO with the conventional \( F_{\text{bif}} \) satisfied both \( F_{\text{bif}} < C_{\text{bif}} \) and \( F_{\text{pp}} < C_{\text{pp}} \) (Table 7). However, although the deemed periodic point \( x_p^* \) composed of the detected saddle-node bifurcation parameters violated \( F_{\text{pp}} < C_{\text{pp}} \),
Figure 4: Detected period-doubling ($P^n$) and saddle-node ($G^n$) bifurcation parameters in the Hénon map. $\mu = -1$ or 1, $n = 5$ or 6. (a) NLPSO results using the conventional $F_{bif}$ described by Eq. (5). (b) NLPSO results using the conditional $F_{bif}^*$ described by Eq. (10).

the NLPSO completed the searching and returned an incorrect result. Consequently, the success rate of the PSO$_{pp}$ was 0% and the detected parameter sets located outside the bifurcation curve (Fig. 4(a)). Clearly, NLPSO using the conventional $F_{bif}$ failed to detect the correct saddle-node bifurcation parameters.

On the other hand, when NLPSO was modified by the conditional $F_{bif}^*$, both the PSO$_{bif}$ and PSO$_{pp}$ detected the bifurcation parameters and correct periodic point with high accuracy. Furthermore, the detected bifurcation parameters constituted an orderly bifurcation curve (Fig. 4(b)). Consequently, the conditional $F_{bif}^*$ described by Eq. (10) enables successful detection of saddle-node bifurcation parameters by NLPSO, without damaging the search for period-doubling bifurcation parameters.

6.3. Neuronal network system

To investigate whether NLPSO effectively detects the bifurcation parameters of highly periodic points, we next consider a discrete neuronal network.
system [16, 17] with five dimension and seven parameters. The system is described as follows:

\[
\begin{align*}
    x_1(k+1) &= \lambda_4 x_1(k) + 2 + \lambda_5 \{g(x_1(k) + x_2(k), 0) - g(x_5(k), 0.5)\}, \\
    x_2(k+1) &= \lambda_1 x_2(k) - \lambda_6 \cdot g(x_1(k) + x_2(k), 0) + \lambda_7, \\
    x_3(k+1) &= \lambda_4 x_3(k) + 2 + \lambda_5 \{g(x_3(k) + x_4(k), 0) - g(x_5(k), 0.5)\}, \\
    x_4(k+1) &= \lambda_1 x_4(k) - \lambda_6 \cdot g(x_3(k) + x_4(k), 0) + \lambda_7, \\
    x_5(k+1) &= \lambda_2 \{g(x_1(k) + x_2(k), \lambda_5) + g(x_3(k) + x_4(k), \lambda_5)\} - x_5(k),
\end{align*}
\]

where \(g(u, \theta)\) is defined as

\[
g(u, \theta) = \frac{1}{1 + \exp(-(u - \theta)/\lambda_3)}. \tag{16}
\]

In Eqs. (15) and (16), \(\mathbf{x} = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5\) is a vector of state variables, and \(\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7) \in \mathbb{R}^7\) is a vector of system parameters. In the bifurcation analysis, we varied \(\lambda_1\) and \(\lambda_2\) and (based on previous work [16, 17]) fixed the other parameters as \(\lambda_3 = 0.1\), \(\lambda_4 = 0.5\), \(\lambda_5 = 15\), \(\lambda_6 = 4\) and \(\lambda_7 = 0.5\). Under these settings, the particle positions of \(\text{PSO}_{\text{bif}}\) and \(\text{PSO}_{\text{pp}}\) were \(\mathbf{z}_b = (z_{b1}, z_{b2}) \equiv \lambda = (\lambda_1, \lambda_2)\) and \(\mathbf{z}_p = (z_{p1}, z_{p2}, z_{p3}, z_{p4}, z_{p5}) \equiv \mathbf{x}_p^* = (x_{p1}^*, x_{p2}^*, x_{p3}^*, x_{p4}^*, x_{p5}^*)\); thus, \(L = 2\) and \(N = 5\). The objective functions of the NLPSO are described in Appendix A.

NLPSO with the conditional objective function \(F_{\text{bif}}^*\) detected the saddle-node bifurcation parameters \((\mu = 1)\) of highly periodic points \((n = 60, 61, \cdots, 68)\). The experimental parameters of the NLPSO were those in Table 1, except that the maximum number of \(\text{PSO}_{\text{bif}}\) iterations was doubled to \(T_{\text{max,bif}} = 600\). The search ranges of the variables are summarized in Table 8.

Table 9 summarizes the mean results of 100 simulations for each periodic number \(n\). As shown in the table, the \(\text{PSO}_{\text{bif}}\) of the NLPSO algorithm successfully detected all periodic points, and the success rate should be further improved by increasing the number of particles. Furthermore, as the \(\text{PSO}_{\text{pp}}\) success rates was consistently 100%, the accuracy of the detected saddle-node bifurcation parameters was guaranteed. Figure 5 is a two-dimensional diagram.
Figure 5: Saddle-node bifurcation parameters ($G^n$) in the neuronal network system, detected by NLPSO cased on $F_{bif}^\mu$. $\mu = 1$. $n = 60, 61, \ldots$, or 68.

7. Conclusions

This paper demonstrated the power of NLPSO in detecting not only period-doubling bifurcation parameters, but also saddle-node bifurcation parameters. We first demonstrated that the conventional objective function in NLPSO gives incorrect saddle-node bifurcation parameters, and we clarified the cause of the
problem. After a minor modification (a simple condition), the objective function guaranteed the accuracy of the periodic points comprising the detected bifurcation parameters. The NLPSO with the conditional objective function detected the period-doubling and saddle-node bifurcation parameters in various systems (the circle map, the Hénon map and a neuronal network system with high periodicity). Moreover, our method requires no careful initialization, gradient information or Lyapunov exponents. By assigning the bifurcation parameters detected by NLPSO as the initial point of a conventional curve tracing, we solved the initial value problem of conventional bifurcation analysis. Therefore, the proposed method can potentially improve the accuracy and simplicity of bifurcation analysis.

In future works, we should apply the proposed method to other bifurcations such as Neimark–Sacker bifurcations, unique bifurcations in composite dynamical systems (such as grazing and border-collision bifurcations), continuous dynamical systems, and noisy systems. We will also accelerate the detection speed by modifying and improving the algorithm.

Appendix A. Application of the conditional objective function to the neuronal network system

On the neuronal network system described by Eq. (15), the conditional objective function of the PSO_{bif}, which searches for the bifurcation parameters, is
defined as

\[ F_{\text{bif}}(z_b) = \begin{cases} \det \left( \frac{\partial f^n}{\partial x_p^r} (x_p^*, z_b) - \mu I_N \right), & \text{if } F_{pp}(x_p^*) < C_{pp} \\ \infty, & \text{otherwise} \end{cases} \]  

(A.1)

\[ = \begin{cases} \det \left( \begin{bmatrix} \frac{\partial f_1}{\partial x_{p_1}^r} & \frac{\partial f_1}{\partial x_{p_2}^r} & \frac{\partial f_1}{\partial x_{p_3}^r} & \frac{\partial f_1}{\partial x_{p_4}^r} & \frac{\partial f_1}{\partial x_{p_5}^r} \\ \frac{\partial f_2}{\partial x_{p_1}^r} & \frac{\partial f_2}{\partial x_{p_2}^r} & \frac{\partial f_2}{\partial x_{p_3}^r} & \frac{\partial f_2}{\partial x_{p_4}^r} & \frac{\partial f_2}{\partial x_{p_5}^r} \\ \frac{\partial f_3}{\partial x_{p_1}^r} & \frac{\partial f_3}{\partial x_{p_2}^r} & \frac{\partial f_3}{\partial x_{p_3}^r} & \frac{\partial f_3}{\partial x_{p_4}^r} & \frac{\partial f_3}{\partial x_{p_5}^r} \\ \frac{\partial f_4}{\partial x_{p_1}^r} & \frac{\partial f_4}{\partial x_{p_2}^r} & \frac{\partial f_4}{\partial x_{p_3}^r} & \frac{\partial f_4}{\partial x_{p_4}^r} & \frac{\partial f_4}{\partial x_{p_5}^r} \\ \frac{\partial f_5}{\partial x_{p_1}^r} & \frac{\partial f_5}{\partial x_{p_2}^r} & \frac{\partial f_5}{\partial x_{p_3}^r} & \frac{\partial f_5}{\partial x_{p_4}^r} & \frac{\partial f_5}{\partial x_{p_5}^r} \end{bmatrix} \right), & \text{if } F_{pp}(x_p^*) < C_{pp}, \\ \infty, & \text{otherwise} \end{cases} \]  

(A.2)
where \( x^*_p = \left( x^*_{p1}, x^*_{p2}, x^*_{p3}, x^*_{p4}, x^*_{p5} \right) \equiv (x_1(0), x_2(0), x_3(0), x_4(0), x_5(0)) \), and

\[
\begin{align*}
\frac{\partial f_1}{\partial x^*_{p1}} &= \lambda_4 + \lambda_5 \cdot dg(x_1(n-1-l) + x_2(n-1-l), 0), \\
\frac{\partial f_1}{\partial x^*_{p2}} &= \lambda_5 \cdot dg(x_1(n-1-l) + x_2(n-1-l), 0), \\
\frac{\partial f_1}{\partial x^*_{p5}} &= -\lambda_5 \cdot dg(x_5(n-1-l), 0.5), \\
\frac{\partial f_2}{\partial x^*_{p1}} &= -\lambda_6 \cdot dg(x_1(n-1-l) + x_2(n-1-l), 0), \\
\frac{\partial f_2}{\partial x^*_{p2}} &= \lambda_1 - \lambda_6 \cdot dg(x_1(n-1-l) + x_2(n-1-l), 0), \\
\frac{\partial f_3}{\partial x^*_{p3}} &= \lambda_4 + \lambda_5 \cdot dg(x_3(n-1-l) + x_4(n-1-l), 0), \\
\frac{\partial f_3}{\partial x^*_{p4}} &= \lambda_5 \cdot dg(x_3(n-1-l) + x_4(n-1-l), 0), \\
\frac{\partial f_3}{\partial x^*_{p5}} &= -\lambda_5 \cdot dg(x_5(n-1-l), 0.5), \\
\frac{\partial f_4}{\partial x^*_{p3}} &= -\lambda_6 \cdot dg(x_3(n-1-l) + x_4(n-1-l), 0), \\
\frac{\partial f_4}{\partial x^*_{p4}} &= \lambda_1 - \lambda_6 \cdot dg(x_3(n-1-l) + x_4(n-1-l), 0), \end{align*}
\]
\[
\frac{\partial f_5}{\partial x_{p_1}^*} = \lambda_1 \cdot dg(x_1(n-1-l) + x_2(n-1-l), \lambda_5) \cdot dg(dg(x_1(n-1-l) + x_2(n-1-l), \lambda_5)
\]
\[\quad + dg(x_3(n-1-l) + x_4(n-1-l), \lambda_5), 0),
\]
\[
\frac{\partial f_5}{\partial x_{p_2}^*} = \frac{\partial f_5}{\partial x_{p_3}^*},
\]
\[
\frac{\partial f_5}{\partial x_{p_3}^*} = \lambda_1 \cdot dg(x_3(n-1-l) + x_4(n-1-l), \lambda_5) \cdot dg(dg(x_1(n-1-l) + x_2(n-1-l), \lambda_5)
\]
\[\quad + dg(x_3(n-1-l) + x_4(n-1-l), \lambda_5), 0),
\]
\[
\frac{\partial f_5}{\partial x_{p_4}^*} = \frac{\partial f_5}{\partial x_{p_3}^*},
\]
\[
\frac{\partial f_5}{\partial x_{p_5}^*} = -\lambda_2,
\]
\[
\frac{\partial f_1}{\partial x_{p_3}^*} = \frac{\partial f_2}{\partial x_{p_4}^*} = \frac{\partial f_2}{\partial x_{p_5}^*} = \frac{\partial f_3}{\partial x_{p_4}^*} = \frac{\partial f_4}{\partial x_{p_4}^*} = \frac{\partial f_4}{\partial x_{p_5}^*} = \frac{\partial f_4}{\partial x_{p_5}^*} = 0
\]
\[
dg(u, \theta) = \frac{\partial g(u, \theta)}{\partial u} = \frac{\exp(-u/\lambda_3)}{\lambda_3 (\exp(-u/\lambda_3) + 1)^2}.
\]

Acknowledgments

The authors are grateful to Y. Tomimura for useful discussions and assistance with the numerical simulations. This work was supported by JSPS KAKENHI Grant Number 16K16124.

References

References


Table 4: Five representative period-doubling ($I^n$) and saddle-node ($G^n$) bifurcation parameters of 2-, 3- and 5-periodic points in the circle map, detected by NLPSO using $F_{\text{bif}}^*$.

<table>
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<th>$\lambda_1$</th>
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<th>$F_{\text{bif}}^*(\lambda)$</th>
<th>$t_{\text{end}}$</th>
<th>$x_p$</th>
<th>$F_{\text{pp}}(x_p)$</th>
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Table 5: Simulation results of the circle map (averaged over 100 independent trials of NLPSO using $F_{bif}^n$).

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<th>PSO$_{pp}$</th>
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Table 6: Search ranges of variables on the Hènon map.

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<th>$\lambda_1$ (= $z_b^1$)</th>
<th>$\lambda_2$ (= $z_b^2$)</th>
<th>($x_{p1}^<em>, x_{p2}^</em>$) (= $z_p^*$)</th>
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<td>[0.08, 0.28]</td>
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Table 7: Simulation results on the Hénon map (averaged over 100 independent trials of NLPSO using conventional $F_{\text{bif}}$ and conditional $F_{\text{bif}}^*$).

<table>
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<tr>
<th>$n$</th>
<th>PSO$_{\text{bif}}$</th>
<th>PSO$_{\text{pp}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>SD</td>
</tr>
<tr>
<td>$F_{\text{bif}}$ or $F_{\text{bif}}^*$</td>
<td>$F_{\text{bif}}$ or $F_{\text{bif}}^*$</td>
<td>$t_{\text{end}}$</td>
</tr>
</tbody>
</table>

**NLPSO using the conventional $F_{\text{bif}}$**

<table>
<thead>
<tr>
<th>$F^n$</th>
<th>5</th>
<th>$4.99 \times 10^{-4}$</th>
<th>$2.92 \times 10^{-4}$</th>
<th>26.4</th>
<th>100</th>
<th>$6.77 \times 10^{-6}$</th>
<th>$2.40 \times 10^{-6}$</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6</td>
<td>$5.01 \times 10^{-4}$</td>
<td>$2.73 \times 10^{-4}$</td>
<td>46.74</td>
<td>100</td>
<td>$7.18 \times 10^{-6}$</td>
<td>$2.10 \times 10^{-6}$</td>
<td>100</td>
</tr>
<tr>
<td>$G^n$</td>
<td>5</td>
<td>$1.22 \times 10^{-8}$</td>
<td>$1.41 \times 10^{-8}$</td>
<td>1</td>
<td>100</td>
<td>$1.92 \times 10^{-2}$</td>
<td>$1.23 \times 10^{-2}$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>$2.97 \times 10^{-6}$</td>
<td>$2.79 \times 10^{-5}$</td>
<td>1.27</td>
<td>100</td>
<td>$1.06 \times 10^{-2}$</td>
<td>$1.11 \times 10^{-2}$</td>
<td>0</td>
</tr>
</tbody>
</table>

**NLPSO using the conditional $F_{\text{bif}}^*$**

<table>
<thead>
<tr>
<th>$F^n$</th>
<th>5</th>
<th>$5.11 \times 10^{-4}$</th>
<th>$2.92 \times 10^{-4}$</th>
<th>26</th>
<th>100</th>
<th>$6.77 \times 10^{-6}$</th>
<th>$2.40 \times 10^{-6}$</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6</td>
<td>$4.31 \times 10^{-4}$</td>
<td>$3.05 \times 10^{-4}$</td>
<td>41.79</td>
<td>100</td>
<td>$6.39 \times 10^{-6}$</td>
<td>$2.41 \times 10^{-6}$</td>
<td>100</td>
</tr>
<tr>
<td>$G^n$</td>
<td>5</td>
<td>$4.71 \times 10^{-4}$</td>
<td>$2.92 \times 10^{-4}$</td>
<td>96.25</td>
<td>100</td>
<td>$7.27 \times 10^{-6}$</td>
<td>$2.61 \times 10^{-6}$</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>$5.41 \times 10^{-4}$</td>
<td>$6.80 \times 10^{-4}$</td>
<td>153.11</td>
<td>96</td>
<td>$7.62 \times 10^{-6}$</td>
<td>$2.30 \times 10^{-6}$</td>
<td>100</td>
</tr>
</tbody>
</table>

Table 8: Search ranges of variables in the neuronal network system.

<table>
<thead>
<tr>
<th>Variable</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$(x_{p1}^<em>, x_{p2}^</em>, x_{p3}^<em>, x_{p4}^</em>)$</th>
<th>$(x_{p5}^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Range</td>
<td>$[0.8905, 0.891]$</td>
<td>$[0.705, 0.74]$</td>
<td>$[-40, 40]^4$</td>
<td>$[-1, 1]$</td>
</tr>
</tbody>
</table>
Table 9: Simulation results of saddle-node bifurcation parameter detection on the neuronal network system (averaged over 100 independent trials of NLPSO using $F^*_{bif}$).

<table>
<thead>
<tr>
<th>n</th>
<th>PSO_{bif}</th>
<th>PSO_{pp}</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$F^*_{bif}$</td>
<td>$F^*_{bif}$</td>
<td>Mean</td>
<td>SD</td>
<td>Mean</td>
<td>Suc[%]</td>
<td>Mean</td>
<td>SD</td>
<td>$F^*_{pp}$</td>
<td>Suc[%]</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>$4.74 \times 10^{-4}$</td>
<td>$3.23 \times 10^{-4}$</td>
<td>152.02</td>
<td>100</td>
<td>9.12 $\times 10^{-6}$</td>
<td>1.07 $\times 10^{-6}$</td>
<td>100</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>61</td>
<td>$4.70 \times 10^{-4}$</td>
<td>$3.25 \times 10^{-4}$</td>
<td>145.59</td>
<td>100</td>
<td>9.25 $\times 10^{-6}$</td>
<td>9.28 $\times 10^{-7}$</td>
<td>100</td>
<td></td>
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</tr>
<tr>
<td>62</td>
<td>$4.98 \times 10^{-4}$</td>
<td>$3.00 \times 10^{-4}$</td>
<td>164.54</td>
<td>100</td>
<td>9.00 $\times 10^{-6}$</td>
<td>1.19 $\times 10^{-6}$</td>
<td>100</td>
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<td></td>
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</tr>
<tr>
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<td>$6.57 \times 10^{-4}$</td>
<td>$7.78 \times 10^{-4}$</td>
<td>210.72</td>
<td>95</td>
<td>9.16 $\times 10^{-6}$</td>
<td>1.00 $\times 10^{-6}$</td>
<td>100</td>
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</tr>
<tr>
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<td>$7.01 \times 10^{-4}$</td>
<td>$8.82 \times 10^{-4}$</td>
<td>217.4</td>
<td>94</td>
<td>8.95 $\times 10^{-6}$</td>
<td>1.09 $\times 10^{-6}$</td>
<td>100</td>
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<tr>
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<td>$6.47 \times 10^{-4}$</td>
<td>$6.75 \times 10^{-4}$</td>
<td>219.61</td>
<td>92</td>
<td>9.21 $\times 10^{-6}$</td>
<td>9.12 $\times 10^{-7}$</td>
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</tr>
<tr>
<td>66</td>
<td>$7.69 \times 10^{-4}$</td>
<td>$1.24 \times 10^{-3}$</td>
<td>241.7</td>
<td>91</td>
<td>9.11 $\times 10^{-6}$</td>
<td>9.90 $\times 10^{-7}$</td>
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<tr>
<td>67</td>
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<td>96</td>
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<td>8.87 $\times 10^{-6}$</td>
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<tr>
<td>68</td>
<td>$6.43 \times 10^{-4}$</td>
<td>$4.44 \times 10^{-4}$</td>
<td>254.03</td>
<td>96</td>
<td>9.20 $\times 10^{-6}$</td>
<td>9.56 $\times 10^{-7}$</td>
<td>100</td>
<td></td>
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</tr>
</tbody>
</table>